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# A Distributed Control Algorithm for Internal Flow Management in a Multi-Zone Climate Unit

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## Abstract

We examine a distributed control problem for internal flow management in a multi-zone climate unit. The problem consists of guaranteeing prescribed indoor climate conditions in a cascade connection of an arbitrarily large number of communicating zones, in which air masses are exchanged to redirect warm air from hot zones (which need to be cooled down) to cold zones (which need to be heated up), and to draw as much fresh air as possible to hot zones, relying on the ventilation capacity of neighboring “collaborative” zones. The controller of each zone must be designed so as to achieve the prescribed climate condition, while fulfilling the constraints imposed by the neighboring zones — due to their willingness to cooperate or not in the air exchange — and the conservation of flow, and despite the action of unknown disturbances. We devise control laws which yield *hybrid* closed-loop systems, depend on local feedback information, take on values in a finite discrete set, and cooperate with neighbor controllers to achieve different compatible control objectives, while avoiding conflicts.

**Keywords** Climate control, distributed control, hybrid systems, nonlinear control, networked control systems.

## Nomenclature

$T_i$	Indoor air temperature of Zone $i$ [ $^{\circ}C$ ]
$T_{amb}$	Temperature of the supplied air [ $^{\circ}C$ ]
$x_i$	Normalized indoor air temperature of Zone $i$ $T_i - T_{amb}$
$Q_{in,i}$	Airflow through inlet of zone $i$ [ $m^3/s$ ]
$Q_{out,i}$	Airflow through outlet of zone $i$ [ $m^3/s$ ]
$Q_{ij}$	Internal airflow from zone $i$ to $j$ [ $m^3/s$ ]
$u_T$	Controlled heat production [ $J/s$ ]
$w_T$	Indoor heat production (disturbance) [ $J/s$ ]
$V_i$	Volume of zone $i$ [ $m^3$ ]
$c_{air}$	Air Heat Capacity [ $J/kg/^{\circ}C$ ]
$\rho_{air}$	Air Density [ $kg/m^3$ ]

## 1 Introduction

Distributed control systems have received considerable attention in the recent years due to progress in information and communication technology. Typically, distributed control systems comprises several subsystems for each one of which a local controller must guarantee the achievement of a control task in cooperation with neighboring subsystems. Sensors and actuators are usually not co-located with the system to control and the sensed or control information must be transmitted through a finite data-rate communication channel. The most common application is in the coordinated motion of mobile agents (see [17, 23, 12, 21] just to cite a few), but other examples arise in power control problems in wireless communication [34], in automated highway systems [30], etc.

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In this paper, we discuss a distributed system which arises in a multi-zone climate control unit. The problem consists of guaranteeing prescribed indoor climate conditions in a building partitioned into communicating zones which exchange air flows. The prescribed climate conditions may differ very much from zone to zone. The ultimate goal is to act on the heating and ventilation devices in such a way that the climate requirement for each zone is reached even when large air masses are being exchanged and time-varying disturbances are present. We are interested in actively causing internal air masses exchange so as to make the heating and ventilation mechanism more efficient. Namely, we aim to achieve an automatic procedure to redirect warm air from hot zones (which need to be cooled down) to cold zones (which need to be heated up), and to draw as much fresh air as possible to hot zones, relying on the ventilation capacity of neighboring “collaborative” zones. See [29] for other distributed problems in multi-zone temperature control. The prescribed climate conditions typically mean that temperature and humidity should evolve within an interval of values (the “thermal region”). The focus of the paper will be on the temperature behavior only, but extensions to include the humidity dynamics are possible, although more involved. We refer the reader to [32], [28], [1] and references therein for recent contributions on the problem of climate control, with special emphasis on agricultural and livestock buildings, which was the initial motivation for the present investigation. There is a large literature on climate control problems. The following additional references have some points of contact with the present contribution. In [3] a second-order model is identified to describe the temperature of an imperfectly ventilated room, and then model predictive control is employed to achieve set-point regulation of the temperature. Temperature control for a multi-zone system is examined in [35], where a decentralized controller is proposed for a linearized model of the system. In [5], a fuzzy logic controller is proposed for a multi-layer model of greenhouse climate control unit. The fuzzy controller is tuned by genetic algorithms and its performance compared with bang-bang and PID controllers. The paper [6] develops a nonlinear adaptive controller for climate control in animal building to deal with nonlinear uncertainty. A hybrid control strategy for a heating/ventilation process modeled by a second-order linear system was employed in [9] to cope with different functioning regimes of the process. See [2] for another study on hybrid control synthesis for heating/ventilation systems. When compared with the papers above, our contribution appears to be the only one which proposes a model-based distributed event-driven control algorithm to achieve the prescribed climate requirements for a large-scale fully nonlinear model of the system.

We are interested in a solution which is suitable for implementation in a networked environment, in which sensors, controllers and actuators may be physically separated. As such, we devise control laws which are event-based, and may require sporadic measurements only. The actuators provide a finite and discrete set of values only. Each controller governs the behavior of a single zone using information from contiguous zones, and cooperate with neighbor controllers to achieve different compatible control objectives and avoid conflicts. As a result of our approach, the overall closed-loop system turns out to be *hybrid* ([33], [13]) and *distributed*. The advantages of having distributed controllers in our case lies in easy implementation, reduced computational burden and limited communication needs among the different components of the system. On the other hand, achieving more global targets, such as the fulfillment of an optimal criterion appears harder. Although extremely important, the design of a distributed controller which in addition is proven to be optimal (for instance, with respect to power consumption) goes beyond the scope of the paper. In recent years, many contributors have focused on networked control systems (see [14] for a recent survey). Some of them have focused on the data rate constraint imposed by the finite bandwidth of the communication channel, for both linear (see e.g. [24, 31, 16, 11, 10, 4]) and nonlinear systems ([20, 8, 25, 7]), others on actuators and sensors scheduling for time-based control of networked control systems (see e.g. [27]), and a few have taken into account both data-rate constraints and decentralized nature of the problem (see e.g. [26], and [19], for a multi-vehicle rendezvous control problem under the effect of quantization).

In order to achieve a controller capable of maintaining the climate conditions within the various thermal regions and at the same time capable of managing the internal flow, we introduce a set of coordinating logic variables ([15]) which express the willingness of each zone to cooperate in the flow exchange, depending on the climate conditions of that zone and the neighboring ones. Then, the controller is designed to solve at each time a game theoretic problem ([15], [22]) aimed to keep the state within the thermal zone despite the action of competitive players, namely thermal disturbances, given the constraints imposed by the neighboring zones, which are due to their willingness to cooperate or not in the air exchange. In addition, other constraints must be fulfilled at any time, namely conservation of flow for each zone. Besides proposing a solution to a novel distributed control problem, in which coordination is achieved while fulfilling algebraic constraints, in the paper the topology of the system is exploited to cope with the large dimension of the problem.

In the next section, the dynamic model is introduced. The design of the controllers is described in Section 3. It is explicitly proven in Section 4 that the proposed controllers guarantee the achievement of the control objective while fulfilling all the constraints (including the flow balance). In Section 5, we illustrate the functioning of the controller for a three-zone climate control unit and conclusions are drawn in Section 6.

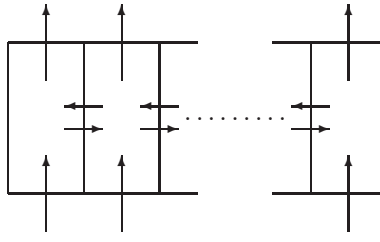


Figure 1: Cascade connection of  $N$  zones.

## 2 System Description and Model

In this paper, we consider a cascade connection of  $N$  rectangular section zones, as illustrated in Figure 1. This corresponds to the arrangement of zones in many real-life situations, such as livestock buildings. However, it appears that the method will work with different arrangements (e.g. those found in cars), provided that the direction and magnitude of the flows to be exchanged among the zones can be set by the actuators. Each zone  $i$ , with  $i \neq 1, N$ , can exchange air with zones  $i - 1$  and  $i + 1$ , while zone 1 and  $N$  can only exchange air with zone 2 and, respectively,  $N - 1$ . For each  $i = 1, \dots, N - 1$ , we denote by  $Q_{i,i+1}$  the amount of air flow exchanged between zone  $i$  and zone  $i + 1$ . More specifically, we have

$$Q_{01} = Q_{10} = 0, \quad Q_{N,N+1} = Q_{N+1,N} = 0, \quad (1)$$

and, for each  $i = 1, \dots, N - 1$ ,

$$Q_{i,i+1} \begin{cases} > 0 & \text{if air flows from } i \text{ to } i + 1 \\ = 0 & \text{otherwise} \end{cases}$$

$$Q_{i+1,i} \begin{cases} > 0 & \text{if air flows from } i + 1 \text{ to } i \\ = 0 & \text{otherwise} . \end{cases}$$

Implicitly, we are assuming that, it is not possible to have simultaneously air exchange from zone  $i$  to zone  $i + 1$  and in the opposite direction. In other words, we assume that

$$Q_{i,i+1} \cdot Q_{i+1,i} = 0 \quad (2)$$

for each  $i = 1, \dots, N$ . Each zone is equipped with an inlet, an outlet, and a ventilation fan, which allow the zone to exchange air with the outside environment and with the neighboring zones. Indeed, by turning on the fan, air is forced out of each zone through the outlet. The amount of air outflow is denoted by the symbol  $Q_{out,i}$ . An amount  $Q_{in,i}$  of inflow enters the zone through the inlet, and the following flow balance must hold: For each  $i = 1, 2, \dots, N$ ,

$$Q_{in,i} + Q_{i-1,i} + Q_{i+1,i} = Q_{out,i} + Q_{i,i-1} + Q_{i,i+1} . \quad (3)$$

We explicitly remark that the amount of inflow depends on the outflow caused by the ventilation fan at the outlet. We now turn our attention to the equations describing the climate condition for each zone. Relevant quantities are the internal temperature  $T_i \in \mathbb{R}$  and humidity  $h_i \in \mathbb{R}_{\geq 0}$ . For the sake of simplicity, in this paper we focus on temperature behavior only, which is therefore taken as state variable. In addition to the ventilation rates  $Q_{out,i}$  provided by the fans, and the inflows  $Q_{in,i}$  flowing through the inlets, another degree of control is given by the heating system, which provides a controlled amount  $u_i$  of heat. Moreover, we shall model the effect of internal disturbances which provide an additional amount of heat  $w_{Ti}$  power. Associated to the air masses which are flowing through the zones is an amount of power proportional to their temperature and the air heat capacity, which gives rise to changes in the temperature inside each zone. By balancing thermal power in each zone, the following equations are easily obtained (cf. e.g. [18], [1]) for  $i = 1, 2, \dots, N$ :

$$\begin{aligned} \rho_{air} c_{air} V_i \frac{d}{dt} T_i &= \rho_{air} c_{air} (Q_{i-1,i} T_{i-1} + Q_{in,i} T_{amb} - Q_{i,i+1} T_i \\ &\quad - Q_{out,i} T_i - Q_{i,i-1} T_i + Q_{i+1,i} T_{i+1}) + u_i + w_{Ti} . \end{aligned} \quad (4)$$

Setting, by a slight abuse of notation,

$$u_i := u_i / (\rho_{air} c_{air}) , \quad w_{Ti} := w_{Ti} / (\rho_{air} c_{air}) ,$$

assuming that outside temperature  $T_{amb}$  is constant and  $T_i > T_{amb}$ <sup>1</sup>, and introducing the change of coordinates

$$x_i = T_i - T_{amb}, \quad i = 1, \dots, N,$$

we obtain, bearing in mind (3), and after easy calculations, the equations, for  $i = 1, 2, \dots, N$ ,

$$V_i \frac{d}{dt} x_i = Q_{i-1,i} x_{i-1} - Q_{i,i+1} x_i - Q_{out,i} x_i - Q_{i,i-1} x_i + Q_{i+1,i} x_{i+1} + u_i + w_{Ti}. \quad (5)$$

In what follows, we shall refer to the  $x_i$ 's simply as the temperature variables, although they differ from the actual temperature variables by a constant offset.

There are limitations on the control effort which can be delivered. In particular, the outflow  $Q_{out,i}$  and the controlled heat must respectively fulfill

$$Q_{out,i} \in [0, Q_{out,i}^M], \quad u_i \in [0, u_i^M], \quad (6)$$

for some known constants  $Q_{out,i}^M$  and  $u_i^M$ . The only way to regulate the amount of inflow is acting on the opening angle of a moving screen at the inlet, which can take only a finite number of positions. As a consequence, we assume that the inflow through the inlets can take only a finite number of values, i.e.

$$Q_{in,i} \in \Delta_i, \quad (7)$$

with  $\Delta_i$  a finite set of nonnegative values which will become clear later (see (14) and the remark following it). We stack in a vector  $U$  all the control signals  $Q_{i,i-1}$ ,  $i = 2, \dots, N$ ,  $Q_{i,i+1}$ ,  $i = 1, \dots, N-1$ ,  $Q_{in,i}$ ,  $Q_{out,i}$ ,  $u_i$ ,  $i = 1, 2, \dots, N$  and denote by  $\mathcal{U}$  the set of admissible (piece-wise continuous) control signals which satisfy (1), (2), (3), (6), (7). Note that not all the components of the vector  $U$  are independent, as they are related through the constraints (2), (3). Additional constraints will be added by the introduction of the coordinating logic variables in the next subsection. Finally, we denote by  $\mathbf{U}$  the set of values taken by the vector  $U$ , letting, for  $i = 1, \dots, N$ ,  $Q_{out,i}$  and  $u_i$  range in the intervals given in (6),  $Q_{in,i}$  take values in the set (7), and  $Q_{i-1,i}$ ,  $Q_{i,i-1}$ ,  $Q_{i,i+1}$ ,  $Q_{i+1,i}$  be such that (2), (3) are fulfilled.

The disturbance signals are not measured, but they are bounded

$$w_{Ti} \in [w_{Ti}^m, w_{Ti}^M], \quad (8)$$

with  $w_{Ti}^m, w_{Ti}^M \in \mathbb{R}$  assumed to be known. The vector

$$W = (w_{T1}, \dots, w_{TN})^T$$

of disturbance signals taking values on the above intervals is said to belong to the class  $\mathcal{W}$  of admissible disturbances. The set

$$\mathbf{W} := [w_{T1}^m, w_{T1}^M] \times \dots \times [w_{TN}^m, w_{TN}^M]$$

denotes the range of values taken by the vector  $W$ .

## 2.1 Coordinating logic variables

Having in mind a cooperative behavior among zones, it is clear that decisions regarding each zone must take into account the behavior of neighboring zones. Furthermore, aiming at a decentralized controller, we would like to implement a controller for each single zone whose strategy is decided on the basis of local information concerning the zone itself and the neighboring zones only. This implies that, in some cases, the objectives for two or more zones can be contrasting and coordination is needed to achieve the overall control strategy. For illustrative purposes, one of these conflicting scenarios is reported below.

*Example.* Consider a 4-zone system with the following scenario: Zone 1 and 3 are cooling down, Zone 2 and 4 are heating up, and the temperatures in the 4 zones satisfy  $x_1 < x_2 < x_3 > x_4$ . As Zone 1 is trying to cool down, it is interested in attracting fresh air from outside. The amount of inflow can be increased if, in addition to the outflow provided by the fan, internal flow from Zone 1 to Zone 2 takes place. Hence, the controller in Zone 1 would be motivated to increase the opening of the inlet 1 so as to let in an amount of air greater than  $Q_{out,1}^M$ . On the other hand, Zone 2 is warming up and the temperature of Zone 3 is higher than the one in Zone 2, whereas the

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<sup>1</sup> Assuming  $T_{amb}$  constant results in no loss of generality, provided that  $T_i$  remains above  $T_{amb}$ . As a matter of fact, the effect of a time-varying ambient temperature can be easily incorporated in the disturbance signal  $w_{Ti}$ . Moreover, the case in which  $T_i \leq T_{amb}$  can be analogously approached, provided that a cooling device – such as a sprinkling system in the livestock building – is included in the model.

temperature in Zone 1 is lower than in Zone 2. Moreover, Zone 3 is cooling down and therefore is interested to let out as much air as possible to the neighboring zones. This implies that Zone 2 will turn on its fan to attract air from Zone 3 (but not from Zone 1). Also notice that Zone 4 is heating up and interested in getting air from Zone 3, since  $x_3 > x_4$ . To avoid the fan in Zone 2 to attract air from Zone 1, the former must signal the latter that the inlet opening at Zone 1 should not allow for an inflow greater than  $Q_{out,1}^M$ , which is clearly in contrast with what Zone 1 is willing to do. At the same time, Zone 2 and 4 must signal Zone 3 they are interested in getting its air, and Zone 3 should acknowledge its willingness to release such air, and correspondingly accommodate its inlet opening.

To systematically resolve conflicts like the one just described, we introduce coordinating logic variables (cf. [15]). Without loss of generality, we regard such variables as state variables which take values in the binary set and whose derivatives are constantly equal to zero. Their values are reset from time to time by the event-based controller to be specified below. For Zone 1, the logic variables are

$$\sigma_{12}^{(1)}, \quad \sigma_{21}^{(1)},$$

for Zone  $N$ ,

$$\sigma_{N-1,N}^{(N)}, \quad \sigma_{N,N-1}^{(N)},$$

and for each zone  $i \neq 1, N$ ,

$$\sigma_{i-1,i}^{(i)}, \quad \sigma_{i,i-1}^{(i)}, \quad \sigma_{i,i+1}^{(i)}, \quad \sigma_{i+1,i}^{(i)}.$$

Each one of the logic variables takes values in the set  $\{0, 1\}$ . If  $\sigma$  is the vector in which all the logic variable are stacked, we have  $\sigma \in \{0, 1\}^{4(N-1)}$ . The logic variables  $\sigma_{i-1,i}^{(i)}, \sigma_{i,i-1}^{(i)}, \sigma_{i,i+1}^{(i)}, \sigma_{i+1,i}^{(i)}$  are set by zone  $i$ . Loosely speaking, if  $\sigma_{i-1,i}^{(i)} = 0$ , this means that zone  $i$  does not want to accept air flow from zone  $i-1$ . On the contrary, if  $\sigma_{i-1,i}^{(i)} = 1$ , the zone is willing to accept air flow from Zone  $i-1$ . Note that  $\sigma_{i-1,i}^{(i)} = 1$  does not necessarily imply that flow will occur from zone  $i-1$  to  $i$ , i.e. not necessarily  $Q_{i-1,i} \neq 0$ , as this depends on whether or not zone  $i-1$  is willing to provide air to zone  $i$ . Similarly for the other logic variables. The rules followed to set the logic variables to a new value and when this should take place is discussed in the next section. Furthermore, for each zone, we introduce ‘‘cumulative’’ variables, which are related to the amount of internal flow that the neighboring zones are willing to exchange in either one of the two directions. Such variables are recursively defined as follows:

$$\begin{aligned} \gamma_N^+ &= 0 \\ \gamma_i^+ &= (\gamma_{i+1}^+ + 1) \sigma_{i,i+1}^{(i)} \cdot \sigma_{i+1,i}^{(i+1)}, \quad i = N-1, \dots, 1, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \gamma_1^- &= 0 \\ \gamma_i^- &= (\gamma_{i-1}^- + 1) \sigma_{i-1,i}^{(i-1)} \cdot \sigma_{i-1,i}^{(i)}, \quad i = 2, \dots, N. \end{aligned} \tag{10}$$

### 3 Design of the Controllers

In this section we introduce the controllers. For the sake of simplicity, first we derive here them under the assumption that the constraints on the flow balance are fulfilled. Then, in the next section, we show that this assumption is actually verified by the controllers. The controllers we are interested in are able to maintain the state of the system within the so-called *thermal region*:

$$F = \Pi_{i=1}^N F_i,$$

where  $F_i = \{x_i : x_i \in [x_i^m, x_i^M]\}$ ,  $x_i^m = T_i^m - T_{amb} \geq 0$ ,  $x_i^M = T_i^M - T_{amb} > x_i^m$ , for all the times, for any initial vector state, and under the action of any admissible disturbance  $W \in \mathcal{W}$ . This kind of controllers are referred to as safety controllers in [22] and our design follows the indications given therein. The controllers here enjoy important features. First, they take into account the constraints imposed by the neighbor zones. In doing so, they are able to guarantee flow exchange among zones when all the zones are willing to carry out this action, while they avoid the raise of conflicts when the actions carried out by neighbor zones are not compatible with each other.

#### 3.1 Design Procedure

The problem is that of designing a controller which guarantees the state  $x_i$  which describes the evolution of the temperature of zone  $i$  to belong to  $F_i$ , the projection on the  $x_i$ -axis of the thermal region  $F$ , for all the times. Following [22], the problem is addressed by formulating the two game problems:

$$\begin{aligned} J_i^{1*}(x, t) &= \max_{U(\cdot) \in \mathcal{U}} \min_{W(\cdot) \in \mathcal{W}} J_i^1(x, U(\cdot), W(\cdot), t), \\ J_i^{2*}(x, t) &= \max_{U(\cdot) \in \mathcal{U}} \min_{W(\cdot) \in \mathcal{W}} J_i^2(x, U(\cdot), W(\cdot), t), \end{aligned} \tag{11}$$

where the value functions

$$\begin{aligned} J_i^1(x, U(\cdot), W(\cdot), t) &= \ell_i^1(x(0)) := x_i(0) - x_i^m, \\ J_i^2(x, U(\cdot), W(\cdot), t) &= \ell_i^2(x(0)) := -x_i(0) + x_i^M, \end{aligned}$$

represent the cost of a trajectory  $x(\cdot)$  which starts from  $x$  at time  $t \leq 0$ , evolves according to the equations (5) under the action of the control  $U(\cdot)$  and the disturbance  $W(\cdot)$ . Clearly,  $F_i = \{x : \ell_i^j(x) \geq 0 \text{ for } j = 1, 2\}$ . In [22], the set of safe sets is defined as  $\{x : J_i^{j*}(x) := \lim_{t \rightarrow -\infty} J_i^{j*}(x, t) \geq 0\}$ , where the function  $J_i^{j*}(x, t)$ ,  $j = 1, 2$ , is found by solving the Hamilton-Jacobi equation

$$\begin{aligned} -\frac{\partial J_i^{j*}(x, t)}{\partial t} &= \min \left\{ 0, H_i^{j*} \left( x, \frac{\partial J_i^{j*}(x, t)}{\partial x} \right) \right\} \\ J_i^{j*}(x, 0) &= \ell_i^j(x), \end{aligned} \quad (12)$$

$H_i^{j*}(x, p)$ , the optimal Hamiltonian, is computed through the point-wise optimization problem

$$H_i^{j*}(x, p) = \max_{U \in \mathbf{U}} \min_{W \in \mathbf{W}} H_i^{j*}(x, p, U, W), \quad (13)$$

and

$$H_i^{j*}(x, p, U, W) = p^T f(x, U, W).$$

Notice that, by (12), at each time  $J_i^{j*}(x, t)$  is non decreasing. Hence, if  $J_i^{j*}(x) \geq 0$ , then  $J_i^{j*}(x, 0) \geq 0$  as well, i.e.  $\ell_i^j(x) \geq 0$ . In other words, as expected, the set of safe states  $\{x : J_i^{j*}(x) \geq 0\}$  is included in the set  $\{x : \ell_i^j(x) \geq 0\}$ .

### 3.2 Controllers

To the purpose of designing the controllers, it is convenient to characterize the maximal controlled invariant set contained in  $F$  ([22]), i.e. the largest set of initial conditions for the state variables for which there exist control actions which maintain the state within  $F$  no matter what the admissible disturbance acting on the system is. The system being bilinear, it is not difficult to show that the maximal controlled invariant set coincides with  $F$  (see Proposition 1). The controllers we design below at any time guarantee the controlled temperature in each zone to be either increasing or decreasing. Given that, for each  $i = 1, \dots, N$ , the local controller has access at any time to the temperatures  $x_{i-1}, x_i, x_{i+1}$  (to  $x_i, x_{i+1}$  if  $i = 1$ , and to  $x_{i-1}, x_i$  if  $i = N$ ), and to the coordinating variables, and that it also knows whether the zone is in the “cooling down” or “heating up” mode, the values for the coordinating logic variables and controls are chosen so as to enforce the maximizing controller  $U(\cdot)$  for the game  $J_i^{1*}(x, t)$ , if the zone is heating up, or for the game  $J_i^{2*}(x, t)$ , if the zone is cooling down, and to take into account the additional constraints imposed by the logic variables of the neighboring zones. In the calculations which lead to the design of the controller, we adopt the notation  $Q_{in,i}^M$  to denote the value <sup>2</sup>.

$$Q_{in,i}^M := Q_{out,i}^M + \sum_{j=1}^{\gamma_i^-} Q_{out,i-j}^M + \sum_{j=1}^{\gamma_i^+} Q_{out,j+i}^M. \quad (14)$$

*Remark.* Depending on the values of  $\gamma_i^-, \gamma_i^+$ , which in turn depend on the values taken by the coordinating logic variables, the variable  $Q_{in,i}^M$  can represent different values. All the possible values for  $Q_{in,i}^M$  obtained from (14) plus the zero value define the set  $\Delta_i$  introduced in Section 2. Note that the  $Q_{in,i}^M$ ’s depend on the values of the maximal fan capacity of all the zones. If such a knowledge is not available, it is possible to avoid it by proper redefinition of the  $\gamma_i^-$ ’s and  $\gamma_i^+$ ’s. We do not pursue it here, as it would require a more cumbersome notation.

We now introduce, for each Zone  $i$ , the controller which is able to handle the conflicting scenarios. To this purpose we need to explicitly take into account the conditions at the neighbor zones, namely temperatures and logic variables. As a result, for each Zone  $i$ , we precisely characterize the optimal controller which satisfies the game problems (11). Furthermore, by construction, whenever the neighbor zones agree on the actions to carry out (and this can be assessed from the values taken by the coordinating logic variables), warm air is redirected from zones which are cooling down to zones which are heating up and are at lower temperatures. At the same time, the zones which are heating up collaborate with the neighbor zones which are cooling down to increase the amount of outflow. The controller is summarized in Table 1. For the special cases  $i = 1, N$ , the controller simplifies, as it can be seen in Table 2 and, respectively, 3. It then becomes very easy to represent the behavior of the switched controller by a graph (see Figures 2, and, respectively, 3).

<sup>2</sup>In the sums, if  $\gamma_i^+ = 0$  ( $\gamma_i^- = 0$ ), then  $\sum_{j=1}^{\gamma_i^+} Q_{out,j+i}^M = 0$  ( $\sum_{j=1}^{\gamma_i^-} Q_{out,i-j}^M = 0$ )



**Proposition 1** *Suppose that*

$$\begin{aligned} Q_{i,i+1} &= \sum_{j=1}^{\gamma_i^+} Q_{out,j+i}^M \quad i = 1, 2, \dots, N-1, \\ Q_{i,i-1} &= \sum_{j=1}^{\gamma_i^-} Q_{out,i-j}^M \quad i = 2, \dots, N-2, N-1, \end{aligned} \tag{15}$$

and let, for each  $i = 1, 2, \dots, N$ ,

$$u_i^M > -w_{Ti}^m, \tag{16}$$

and

$$Q_{out,i}^M x_i^M - w_{Ti}^M > 0, \tag{17}$$

hold. Then, for each  $i = 1, 2, \dots, N$ , the maximal controlled invariant set of  $F_i$  coincides with  $F_i$  itself, the controller described in Table 1 if  $i \neq 1, N$  (respectively, in Table 2 if  $i = 1$  and in Table 3 if  $i = N$ ), renders  $F_i$  invariant and is the maximizing controller of the game problems (11).

*Proof.* See Appendix A.

*Remark.* A few observations are in order:

- Loosely speaking, the condition (15) amounts to require the flow balance fulfilled for each zone. We shall verify in the next section that this results in no loss of generality, for the controller is actually capable of guaranteeing the fulfillment of such constraint.
- Non strict inequalities in (16) and (17) suffice for  $F_i$  to be the maximal controlled invariant set. However, having them fulfilled with strict inequalities guarantees the temperature of each zone to be either decreasing or increasing.
- A few additional words about conditions (16) and (17) are in order. Although they may appear restrictive, they are very frequently encountered in practice. In fact, for many applications, the disturbance  $w_{Ti}^M$  is nonnegative, that is thermal dispersion is largely dominated by internally generated disturbance heat (this is especially true in livestock buildings or otherwise overcrowded closed environments). Hence, (16) is already satisfied with  $u_i^M = 0$ . (Notice that in this case, having a heating strategy still makes sense, in order to rapidly steer the temperature to a large enough value at which it is safe to let in fresh air from outside to increase the indoor air quality.) Furthermore, we shall see in the next section that it is not always possible to use warm air originated from neighbor zones to heat up the temperature of a zone. This is the case for instance when neighbor zones are themselves heating up. In such cases, the zone must be able to heat up only relying on its own heating system, and therefore condition (16) is necessary for the problem to be feasible.

On the other hand, having exceedingly high indoor temperature can be harmful and must be avoided at any cost. Hence, it is mandatory to equip a room with a fan which is able to steer away the temperature from the dangerous limit, so that (17) is satisfied. Notice that, as for condition (16), even for condition (17) a smaller safety set does not help to relax these requirements. Indeed, (16) is independent of the state, while (17) is such that if it holds for any  $x_i$  which is inside the thermal region, then it is *a fortiori* true for  $x_i = x_i^M$ .

In the next section, we show that the controllers introduced here are actually feasible controllers, meaning that the flow balance (3) is fulfilled for each zone. In other words, it will become clear that the condition (15) is actually guaranteed by our design of the controller.

## 4 Feasibility of the controllers

The main obstacle to prove the feasibility of the controllers investigated in the previous section comes from the fact that the dynamics of each zone is closely intertwined with those of the neighbor zones and that the number of zones are arbitrarily large. Nevertheless, we can exploit the topology of the system, namely the configuration according to which the zones are positioned, to approach the problem by an inductive argument. In particular, we



Mode		$Q_{out,i}$	$Q_{in,i}$	$u_i$	$\sigma_{i-1,i}^{(i)}$	$\sigma_{i,i-1}^{(i)}$	$\sigma_{i,i+1}^{(i)}$	$\sigma_{i+1,i}^{(i)}$
	<i>Zone i Cooling Down</i>							
1	Under any condition	$Q_{out,i}^M$	$Q_{in,i}^M$	0	0	1	1	0
	<i>Zone i Heating Up</i>							
2	$x_{i-1} \leq x_i \wedge x_i \geq x_{i+1}$	0	0	$u_i^M$	0	0	0	0
3	$x_{i-1} > x_i \wedge x_i \geq x_{i+1}$ $\sigma_{i-1,i}^{(i-1)} = 0$	0	0	$u_i^M$	1	0	0	0
4	$x_{i-1} > x_i \wedge x_i \geq x_{i+1}$ $\sigma_{i-1,i}^{(i-1)} = 1$	$Q_{out,i}^M$	0	$u_i^M$	1	0	1	0
5	$x_{i-1} \leq x_i \wedge x_i < x_{i+1}$ $\sigma_{i+1,i}^{(i+1)} = 0$	0	0	$u_i^M$	0	0	0	1
6	$x_{i-1} \leq x_i \wedge x_i < x_{i+1}$ $\sigma_{i+1,i}^{(i+1)} = 1$	$Q_{out,i}^M$	0	$u_i^M$	0	1	0	1
7	$x_{i-1} > x_i \wedge x_i < x_{i+1}$ $\sigma_{i-1,i}^{(i-1)} = 0 \wedge \sigma_{i+1,i}^{(i+1)} = 0$	0	0	$u_i^M$	1	0	0	1
8	$x_{i-1} > x_i \wedge x_i < x_{i+1}$ $\sigma_{i-1,i}^{(i-1)} = 1 \wedge \sigma_{i+1,i}^{(i+1)} = 0$	$Q_{out,i}^M$	0	$u_i^M$	1	0	1	0
9	$x_{i-1} > x_i \wedge x_i < x_{i+1}$ $\sigma_{i-1,i}^{(i-1)} = 0 \wedge \sigma_{i+1,i}^{(i+1)} = 1$	$Q_{out,i}^M$	0	$u_i^M$	0	1	0	1
10	$x_{i-1} > x_i \wedge x_i < x_{i+1} \wedge x_{i-1} \geq x_{i+1}$ $\sigma_{i-1,i}^{(i-1)} = 1 \wedge \sigma_{i+1,i}^{(i+1)} = 1$	$Q_{out,i}^M$	0	$u_i^M$	1	0	1	0
11	$x_{i-1} > x_i \wedge x_i < x_{i+1} \wedge x_{i-1} < x_{i+1}$ $\sigma_{i-1,i}^{(i-1)} = 1 \wedge \sigma_{i+1,i}^{(i+1)} = 1$	$Q_{out,i}^M$	0	$u_i^M$	0	1	0	1

Table 1: Summary of the control law for Zone  $i$ . Transitions from the “Cooling Down” mode to the “Heating Up” modes are triggered only if the clause  $x_i \geq x_i^M$  is verified. Similarly, converse transitions occur only if the clause  $x_i \leq x_i^m$  holds true.

Mode		$Q_{out,1}$	$Q_{in,1}$	$u_1$	$\sigma_{12}^{(1)}$	$\sigma_{21}^{(1)}$
	<i>Zone 1 Cooling Down</i>					
1	Under any condition	$Q_{out,1}^M$	$Q_{in,1}^M$	0	1	0
	<i>Zone 1 Heating Up</i>					
2	$x_1 \geq x_2$	0	0	$u_1^M$	0	0
3	$x_1 < x_2$ $\sigma_{21}^{(2)} = 0$	0	0	$u_1^M$	0	1
4	$x_1 < x_2$ $\sigma_{21}^{(2)} = 1$	$Q_{out,1}^M$	0	$u_1^M$	0	1

Table 2: Summary of the control law for Zone 1.

Mode		$Q_{out,N}$	$Q_{in,N}$	$u_N$	$\sigma_{N-1,N}^{(N)}$	$\sigma_{N,N-1}^{(N)}$
	<i>Zone N Cooling Down</i>					
1	Under any condition	$Q_{out,N}^M$	$Q_{in,N}^M$	0	0	1
	<i>Zone N Heating Up</i>					
2	$x_{N-1} < x_N$	0	0	$u_N^M$	0	0
3	$x_{N-1} \geq x_N$ $\sigma_{N-1,N}^{(N-1)} = 0$	0	0	$u_N^M$	1	0
4	$x_{N-1} \geq x_N$ $\sigma_{N-1,N}^{(N-1)} = 1$	$Q_{out,N}^M$	0	$u_N^M$	1	0

Table 3: Summary of the control law for Zone  $N$ .

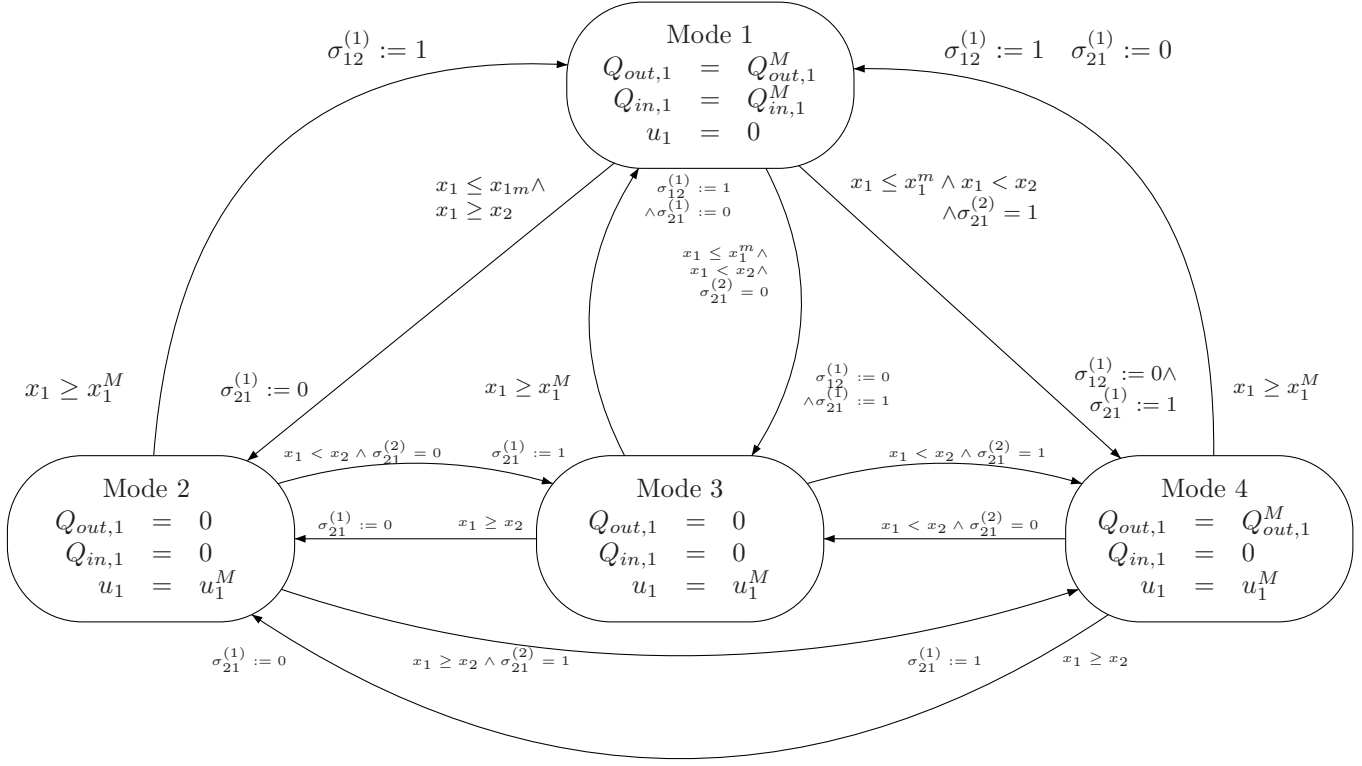


Figure 2: The graph represents the event-based controller for Zone 1. When the edge has two labels the first one represents the guard, the second one (denoted by  $:=$ ) the reset. When no reset is present, then all the variables remain unchanged upon the transition. All the guards at the edges linking the three states at the bottom should include the clause  $\neg x_1 \geq x_1^M$ , which is omitted for the sake of simplicity. For each mode (discrete state) only three representative values are indicated. The remaining values can be derived from Table 2 and allow to obtain the continuous-time model associated with the discrete state. Thus, for instance, if the zone is in Mode 1, then it is evolving according to the equation  $V_1 \dot{x}_1 = -\sum_{j=1}^{\gamma_1^+} Q_{out,j+1}^M x_1 - Q_{out,1}^M x_1 + w_{T1}$ , with  $\gamma_1^+ = (\gamma_2^+ + 1) \sigma_{12}^{(1)} \sigma_{12}^{(2)}$ , and  $\sigma_{12}^{(1)} = 1, \sigma_{21}^{(1)} = 0$ .

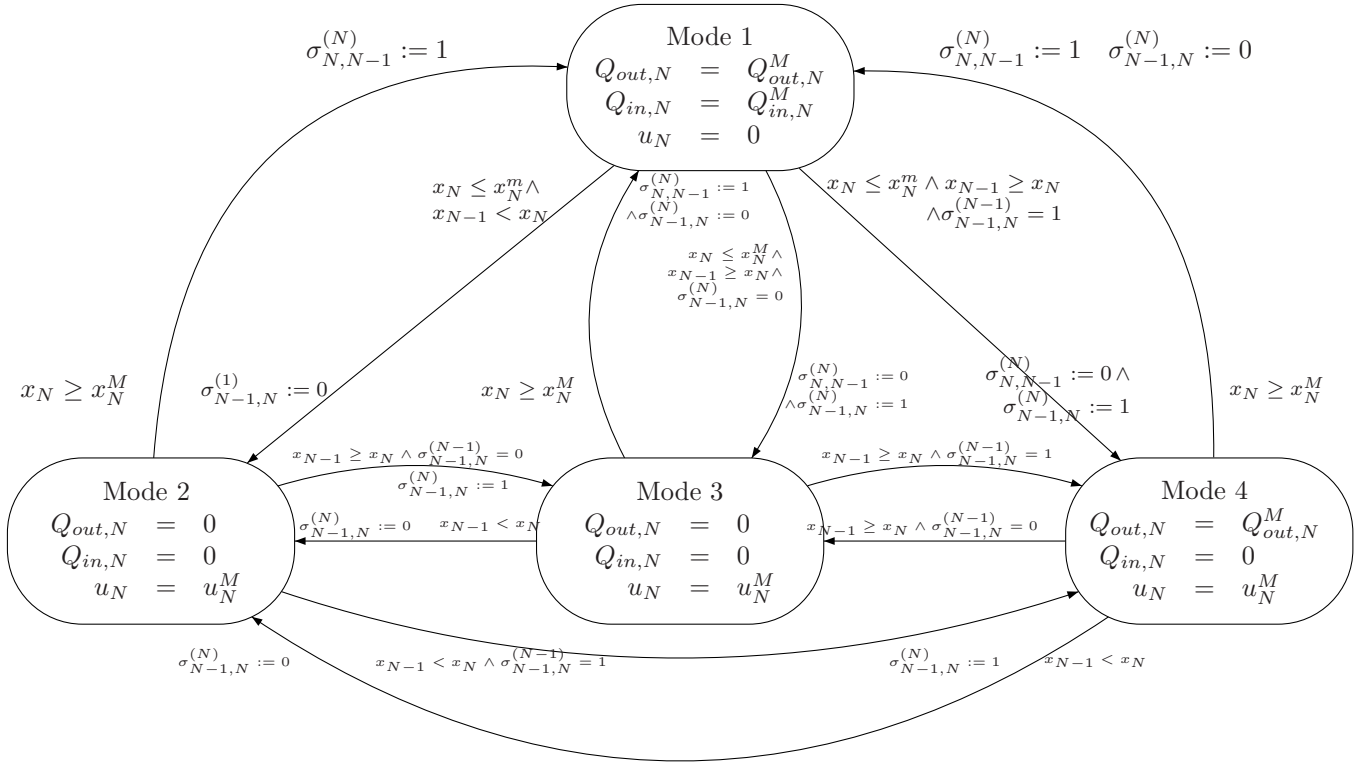


Figure 3: The event-based controller for Zone  $N$ .

shall characterize conditions under which the flow balance is fulfilled for the first 2 zones. Then we shall proceed by showing the conditions under which, assuming that the flow balance is fulfilled up to Zone  $i$ , the flow balance is fulfilled even for Zone  $i + 1$ , and concluding the argument considering the zones  $N - 1$  and  $N$ .

To make the statements below more concise we introduce the following definition:

*Definition.* Let  $i$  be an integer such that  $2 \leq i \leq N - 1$ . Zones  $1, 2, \dots, i$  are said to *conditionally* satisfy the flow balance (3), if (3) is fulfilled for Zones  $1, 2, \dots, i$  provided that:

- If Zone  $i$  is in Mode 1, 4 or 8,

$$Q_{i,i+1} = \sum_{j=1}^{\gamma_i^+} Q_{out,j+i}^M, \quad (18)$$

with  $\gamma_i^+ = (\gamma_{i+1}^+ + 1)\sigma_{i,i+1}^{(i+1)}$ , and  $Q_{i+1,i} = 0$ .

- If Zone  $i$  is in Mode 2, 3, 5, 7 or 10,

$$Q_{i,i+1} = 0 \text{ and } Q_{i+1,i} = 0. \quad (19)$$

- If Zone  $i$  is in Mode 6, 9 or 11,  $Q_{i,i+1} = 0$  and

$$Q_{i+1,i} = \sum_{j=1}^{\gamma_{i+1}^-} Q_{out,i+1-j}^M, \quad (20)$$

with  $\gamma_{i+1}^- = \gamma_i^- + 1$ .

In other words, Zones  $1, 2, \dots, i$  are said to *conditionally* satisfy the flow balance if all of them fulfill the flow balance provided that suitable conditions are met involving only the “terminal” Zone  $i$ .

In the following statements, we consider the  $N$ -zone climate control unit (5), (2), (3), (6), (8), under the action of the controllers introduced above. Proofs are postponed to the Appendices.

**Lemma 1** *Zones 1 and 2 conditionally satisfy the flow balance (3).*

*Proof.* See Appendix B.

Then along the lines of the proof of the previous lemma, the following statement can be proven:

**Lemma 2** *For some integer  $2 \leq i \leq N - 2$ , if Zones  $1, 2, \dots, i$  conditionally satisfy the flow balance (3), then also Zones  $1, 2, \dots, i, i + 1$  conditionally satisfy the flow balance (3).*

*Proof.* See Appendix C.

The arguments above can be finalized to prove that the flow balance is fulfilled for all the zones:

**Proposition 2** *Zones  $1, 2, \dots, N - 1, N$  satisfy the flow balance (3).*

*Proof.* Zones 1 and 2 conditionally satisfy the flow balance (3) by Lemma 1. Then applying repeatedly Lemma 2, one can prove that actually Zones 1 to  $N-1$  conditionally satisfy the flow balance. This means that, the flow balance is fulfilled for Zones 1 to  $N-1$ , provided that conditions (18)-(20) hold with  $i = N - 1$ . Below we prove that such conditions are actually true, and that consequently the flow balance is fulfilled even for Zone  $N$ . Suppose that Zone  $N - 1$  is in Mode 1, 4 or 8. Then, we would like to prove

$$Q_{N-1,N} = \sum_{j=1}^{\gamma_{N-1}^+} Q_{out,j+N-1}^M,$$

with  $\gamma_{N-1}^+ = (\gamma_N^+ + 1)\sigma_{N-1,N}^{(N)}$ , and  $Q_{N,N-1} = 0$ . Bearing in mind the control laws for Zones  $N - 1$  and  $N$ , we can argue in the following way. When Zone  $N$  is in Mode 1 and 2,  $\sigma_{N-1,N}^{(N)} = 0$  and we must verify that  $Q_{N-1,N} = 0$ . In Mode 1,  $Q_{out,N} = Q_{out,N}^M$  and  $Q_{in,N} = Q_{in,N}^M$ . But, Zone  $N - 1$  being in Mode 1, 4 or 8 yields that  $\sigma_{N,N-1}^{(N-1)} = 0$  (see Table 1), and therefore  $Q_{in,N}^M = Q_{out,N}^M$ , which allows to conclude  $Q_{N-1,N} = 0$  and the fulfillment of the flow balance for Zone  $N$ . In Mode 2,  $Q_{out,N} = Q_{in,N} = 0$ , and again  $Q_{N-1,N} = 0$  and the fulfillment of the flow balance

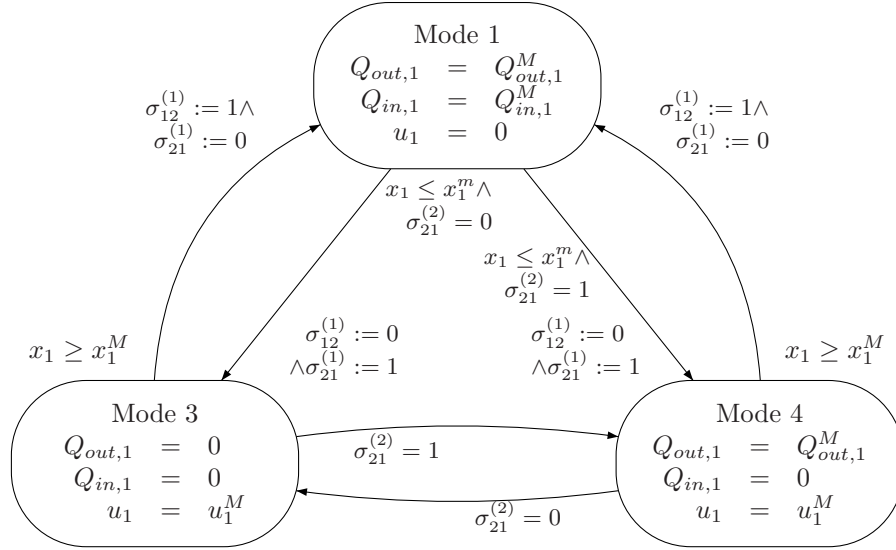


Figure 4: Controller for Zone 1 when  $x_1^M < x_2^m$ .

in Zone  $N$  are trivially true. When Zone  $N$  is in Mode 3 and 4,  $\sigma_{N-1,N}^{(N)} = 1$ , and we must have  $Q_{N-1,N} = Q_{out,N}^M$ . It is not possible to have Zone  $N$  in Mode 3 and Zone  $N-1$  in either one of Modes 1, 4 and 8 because of the contrasting requirements on the logic variables  $\sigma_{N,N-1}^{(N-1)}$ ,  $\sigma_{N-1,N}^{(N-1)}$  by the two Zones. On the other hand, in Mode 4, that  $Q_{N-1,N} = Q_{out,N}^M$  is immediately verified because  $Q_{out,N} = Q_{out,N}^M$  and  $Q_{in,N} = 0$ .

Suppose now that Zone  $N-1$  is in Mode 2, 3, 5, 7 or 10. Then the flow balance for Zone  $N$  must be verified with  $Q_{N-1,N} = Q_{N,N-1} = 0$ . This is immediate if Zone  $N$  is in Mode 2 or 3. Suppose now that Zone  $N$  is in Mode 1, that is  $Q_{out,N} = Q_{out,N}^M$  and  $Q_{in,N} = Q_{in,N}^M$ . Since  $\gamma_N^- = (\gamma_{N-1}^- + 1)\sigma_{N,N-1}^{(N-1)} \cdot 1$ ,  $\gamma_N^- = 0$  when Zone  $N-1$  is in Mode 2, 3 or 10, so that  $Q_{in,N}^M = Q_{out,N}^M$  and the flow balance is proven. On the other hand, the cases in which Zone  $N-1$  is in Mode 5 or 7 and Zone  $N$  is in Mode 1 are not feasible because of  $\sigma_{N-1,N}^{(N)}$ , which is required to be 0 by Zone  $N-1$  in Modes 5 or 7, and imposed to be 1 by Zone  $N$  in Mode 1. Suppose now that Zone  $N$  is in Mode 4. This occurs if  $\sigma_{N-1,N}^{(N-1)} = 1$ , which excludes the possibility for Zone  $N-1$  to be in Mode 2, 3, 5, 7. Suppose then that Zone  $N-1$  is in Mode 10. But neither this case is possible because we should have  $x_{N-1} < x_N$ , while Zone  $N$  in Mode 4 requires  $x_{N-1} \geq x_N$ .

Finally, consider the case that Zone  $N-1$  is in Mode 6, 9 or 11. As this requires  $\sigma_{N,N-1}^{(N)} = 1$ , Zone  $N$  can only operate in Mode 1, as the other modes impose  $\sigma_{N,N-1}^{(N)} = 0$ . Furthermore,  $\gamma_N^- = \gamma_{N-1}^- + 1$ . This implies

$$Q_{in,N} = Q_{out,N}^M + \sum_{j=1}^{\gamma_{N-1}^- + 1} Q_{out,N-j}^M.$$

As  $Q_{out,N} = Q_{out,N}^M$ , then  $Q_{N,N-1} = \sum_{j=1}^{\gamma_{N-1}^- + 1} Q_{out,N-j}^M$ , and therefore the flow balance is satisfied for Zone  $N$ . This ends the proof.

## 5 Numerical results for a 3-zone climate control unit

In this section we present the outcome of a simulation for the 3-zone case, for which the *thermal regions* are defined as  $T_{1m} = T_{3m} = 12^\circ\text{C}$ ,  $T_{1M} = T_{3M} = 14^\circ\text{C}$ ,  $T_{2m} = 14.5^\circ\text{C}$  and  $T_{2M} = 16.5^\circ\text{C}$ . For the convenience of the reader, the simulation is performed in the absolute coordinate system  $T_i = x_i - T^{amb}$ . The thermal regions picked for the simulation imply that whenever Zone 2 is cooling down and either Zone 1 or Zone 3 is heating up internal flow should occur. We also note that, under the present scenario, the controllers take the form illustrated in Figures 4-6, which point out the event-based nature of the design. The zone volumes are set to  $V_1 = 2000\text{ m}^3$  and  $V_2 = V_3 = 1800\text{ m}^3$ , the maximum heating capacities to  $u_1^M = u_2^M = 2^\circ\text{C m}^3\text{ s}^{-1}$  and  $u_3^M = 3^\circ\text{C m}^3\text{ s}^{-1}$  and the

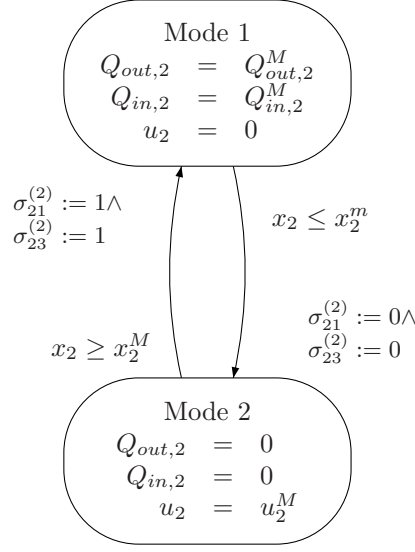


Figure 5: Controller for Zone 2 when  $x_1^M < x_2^m > x_3^M$ .

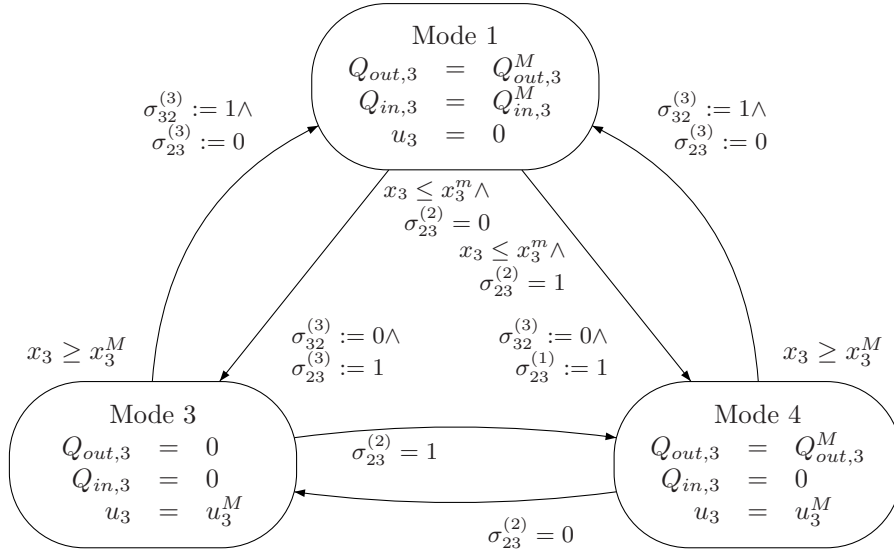


Figure 6: Controller for Zone 3 when  $x_3^M < x_2^m$ .

fan capacities to  $Q_{1,out}^M = 1.3 \text{ m}^3 \text{ s}^{-1}$ ,  $Q_{2,out}^M = 0.9 \text{ m}^3 \text{ s}^{-1}$  and  $Q_{3,out}^M = 1.2 \text{ m}^3 \text{ s}^{-1}$ . The disturbances as well as the ambient temperature are piecewise constant functions of time. The initial state is set to  $T = [14^\circ\text{C} \ 16^\circ\text{C} \ 12.5^\circ\text{C}]^T$ . Figure 7 illustrates the evolution of the three temperatures. Figure 8 illustrates the modes at which each controller is operating the considered time horizon. When Zone 2 is in Mode 1 and Zone 1 and/or 3 are in Mode 4, internal flow occurs. The occurrence of internal flow is depicted in Figure 9. To further point out the use of internal flow, Figure 10 illustrates the time profile for  $Q_{2,in}$  and  $Q_{2,out}$ . While  $Q_{2,out}$  is constantly equal to  $Q_{2,out}^M = 1.9$  when nonzero,  $Q_{2,in}^M$  depends on the cumulative flow variables  $\gamma_2^+$  and  $\gamma_2^-$  and may take 5 different values.

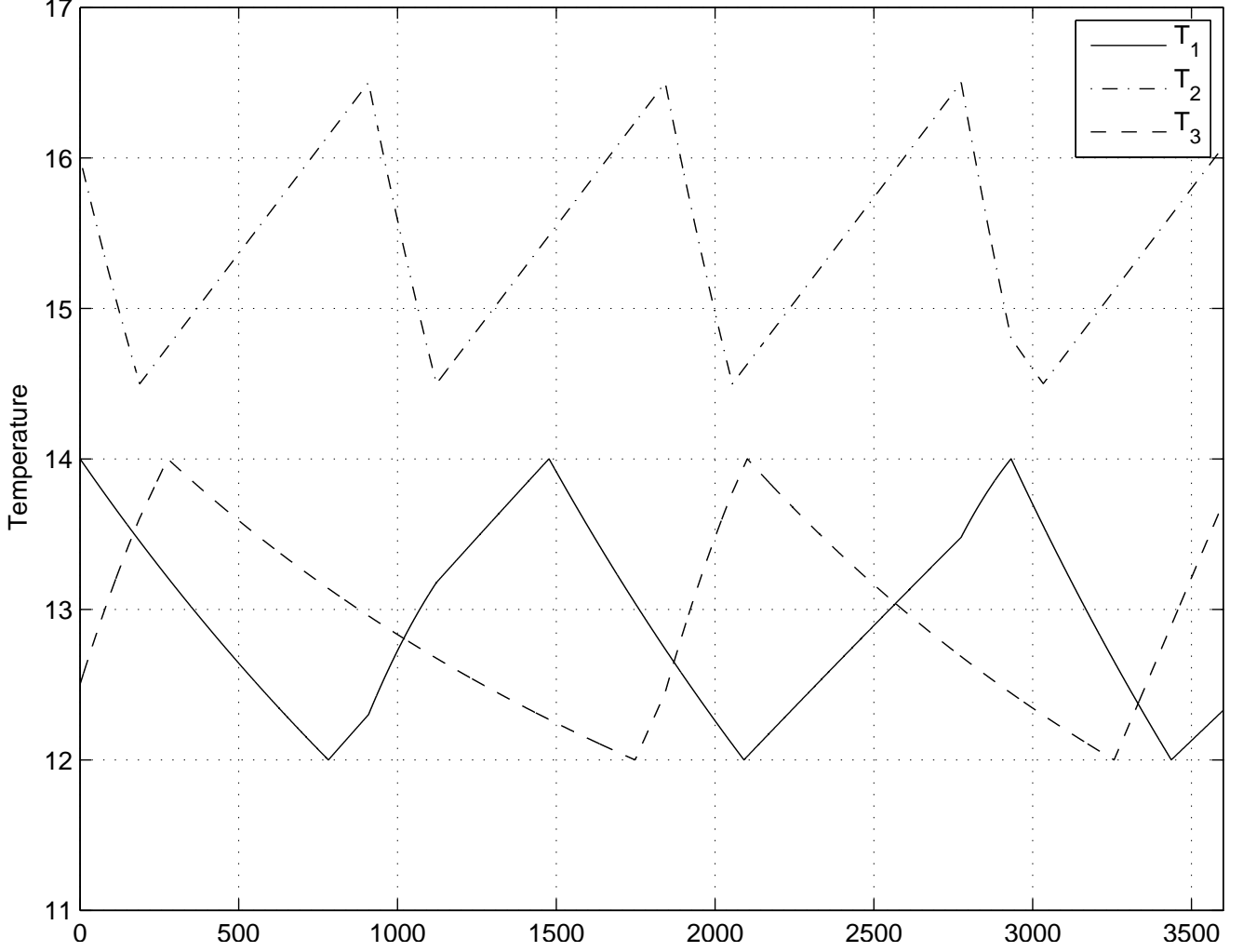


Figure 7: Time history of the three temperatures.

## 6 Conclusion

The paper has discussed a control strategy for a multi zone climate unit capable of maintaining the state within a prescribed (safe) set, by managing the internal flow between zones. The control laws are inherently event-based and distributed, namely they only change action when certain boundaries are met and/or when neighboring conditions change, and additionally they only require feedback information originated from neighboring zones. Our motivation for considering the devised control strategy is the possible implementation in a resource constrained environment using wireless battery powered climatic sensors. Hence, we were after a solution to the problem which allowed to reach the control goal by transmitting feedback information only sporadically. We have observed that the controllers take on values in a finite set, thus allowing for a potentially robust information transmission encoded using a finite number of bits. We have showed that the control law handles internal flow efficiently by using warm air or additional ventilation from neighbor zones to heat up or, respectively, cool down, whenever certain conditions are met. The proposed controller works for an arbitrarily large number of zones. The main feature to overcome the complexity



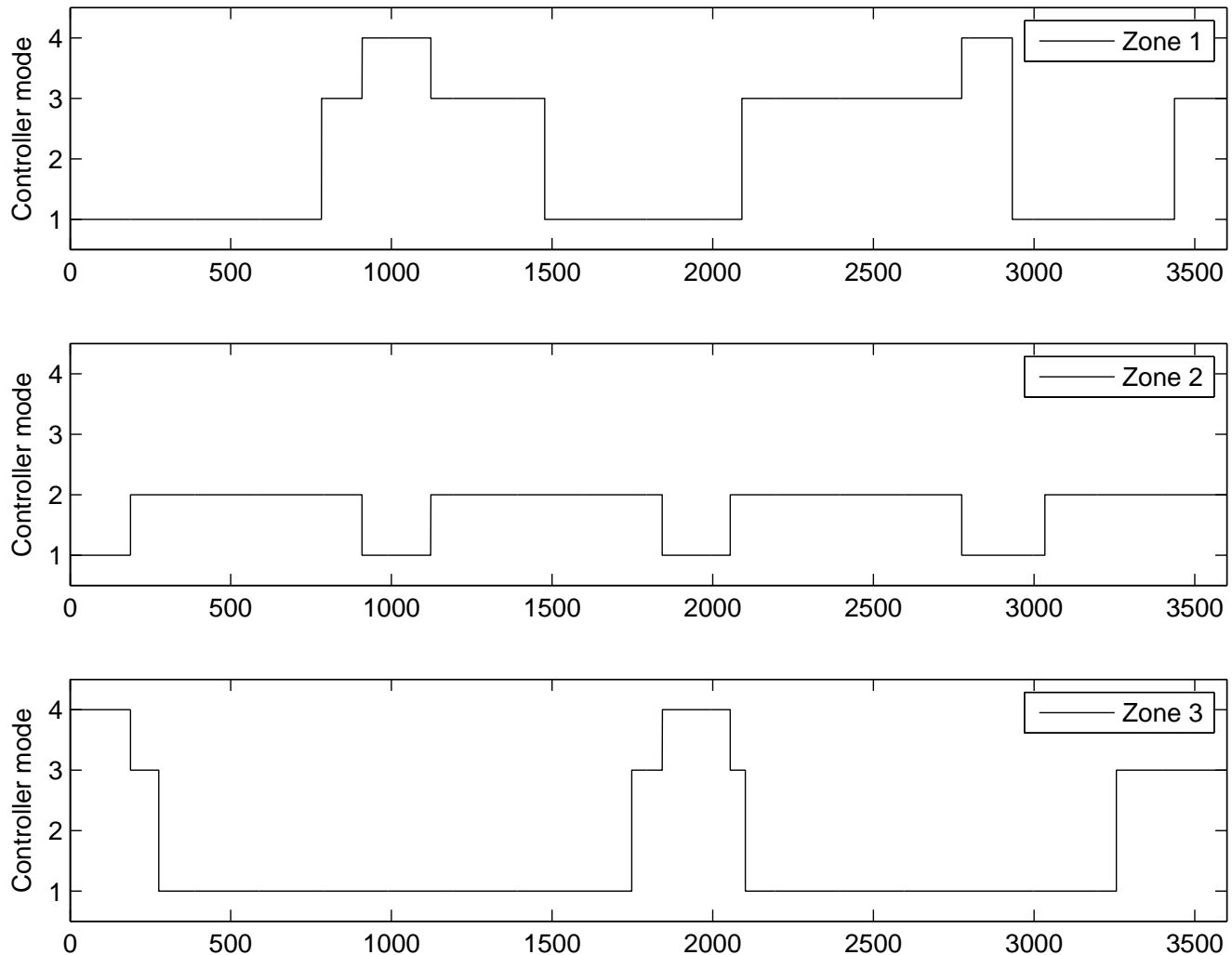


Figure 8: Evolution of the controller modes for the three controllers.

of the analysis due to the large-scale nature of the problem consists of exploiting the topology of the system to show that the analysis and design of an  $N$ -zone system can always be reduced to the 3-zone case. The method is tailored to interconnected systems, with a time-invariant graph underlying the connections, and described by bilinear differential equations with linear algebraic constraints. This appears to be a framework common to other technological fields. In this respect, we believe that the same approach could prove useful even for other applications with more complex topologies and modeled by differential algebraic equations. On the other end, the system we present could be employed to test other design techniques for networked control systems, or to propose other problems of coordinated control. Finally, it could be interesting to investigate the trade-off between more demanding control objectives, such as set-point regulation, and the complexity of the controller, measured in terms of required data rate of the transmission channel and computational capability of the controller.

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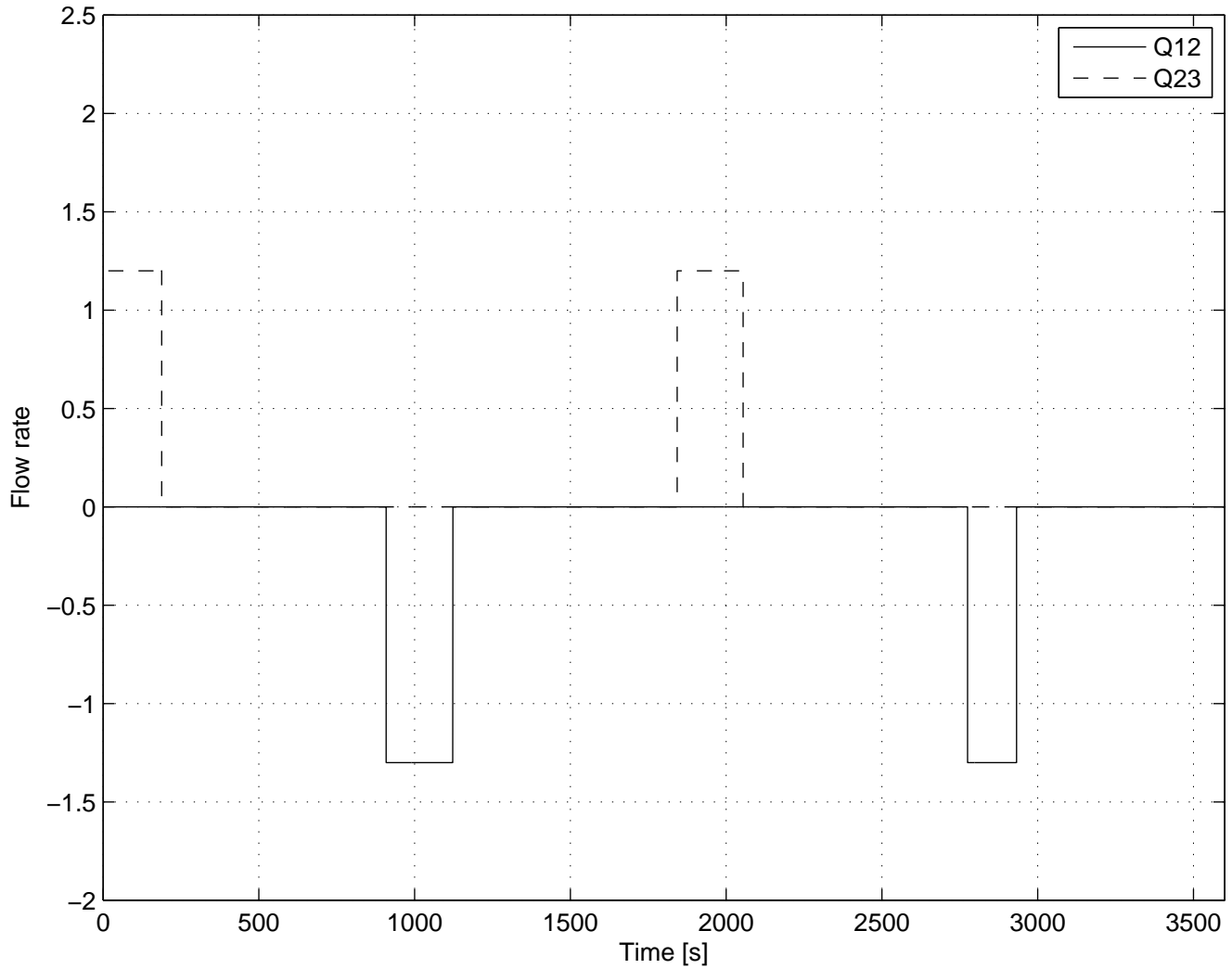


Figure 9: Time history for the internal flow in Zone 2.

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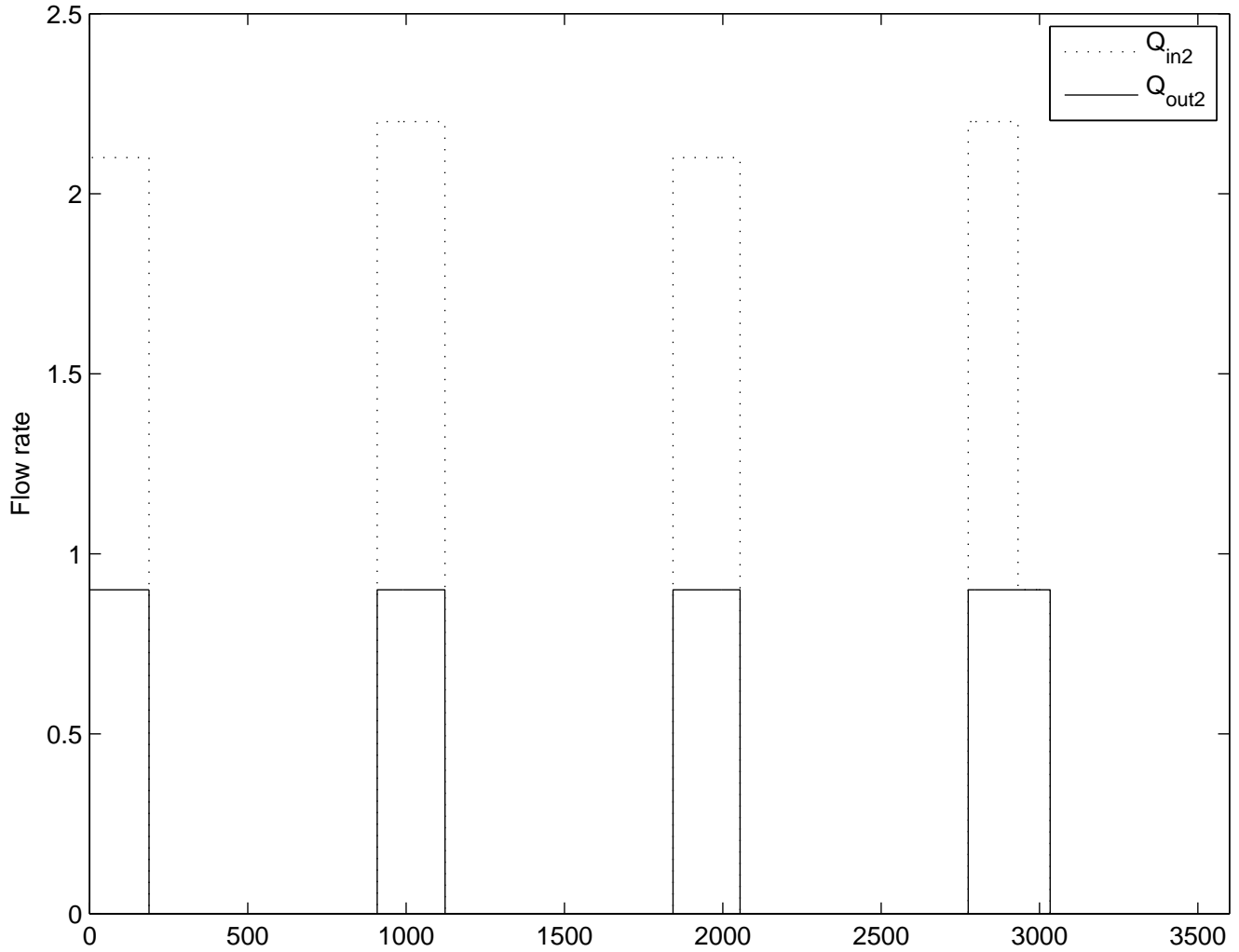


Figure 10: Time history for the inflow and outflow in Zone 2.

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## A Proof of Proposition 1

The proof of the first part of the proposition, namely that  $F_i$  coincides with its maximal controlled invariant set, is shown by the following two simple lemma.

**Lemma 3** *For any  $i = 1, 2, \dots, N$ , if*

$$u_i^M \geq -w_{Tim} , \quad (21)$$

*we have*

$$\{x : J_i^{1*}(x) \geq 0\} = \{x : \ell_i^1(x) \geq 0\} . \quad (22)$$

*Proof.* Let first  $i$  be  $\neq 1, N$ . It is immediately seen that, by (3), the objective function of the static game

$$H_i^{1*} \left( x, \frac{\partial J_i^{1*}(x, 0)}{\partial x} \right)$$

takes the form

$$Q_{i-1,i}(x_{i-1} - x_i) + Q_{i+1,i}(x_{i+1} - x_i) - Q_{in,i}x_i + u_i + w_{Ti} . \quad (23)$$

This shows that

$$H_i^{1*} \left( x, \frac{\partial J_i^{1*}(x, 0)}{\partial x} \right) \geq u_i^M + w_{Ti}^m . \quad (24)$$

In fact, the minimizing  $w$  clearly requires  $w_{Ti} = w_{Ti}^m$ , whereas the maximizing  $U$  imposes  $Q_{in,i} = 0$  and  $u_i = u_i^M$ , being  $x_i \geq 0$ . Furthermore, if  $x_{i-1} - x_i \geq 0$  (respectively,  $x_{i+1} - x_i \geq 0$ ), then the maximizing  $U$  yields  $Q_{i-1,i} \geq 0$  ( $Q_{i+1,i} \geq 0$ ), otherwise  $Q_{i-1,i} = 0$  ( $Q_{i+1,i} = 0$ ). In any case,

$$Q_{i-1,i}(x_{i-1} - x_i) + Q_{i+1,i}(x_{i+1} - x_i) \geq 0 ,$$

and this shows (24). Since  $u_i^M + w_{Ti}^m \geq 0$ , we have

$$H_i^{1*} \left( x, \frac{\partial J_i^{1*}(x, 0)}{\partial x} \right) \geq 0 .$$

One concludes that  $J_i^{1*}(x, 0) \geq 0$  implies  $J_i^{1*}(x, t) \geq 0$  for all  $t$  by (12), and hence  $J_i^{1*}(x) \geq 0$ , i.e.  $\{x : J_i^{1*}(x) \geq 0\} \supseteq \{x : x_i - x_i^m \geq 0\}$ . This proves the thesis for  $i \neq 1, N$ . In the case  $i = 1$  (respectively,  $i = N$ ), (23) becomes

$$Q_{21}(x_2 - x_1) - Q_{in,1}x_1 + u_1 + w_{T1} \quad (Q_{N-1,N}(x_{N-1} - x_N) - Q_{in,N}x_N + u_N + w_N)$$

and simple arguments as before show that (24) holds even with  $i = 1, N$ . The rest of the arguments go through exactly as in the case  $i \neq 1, N$ .

**Lemma 4** For any  $i = 1, 2, \dots, N$ , if

$$Q_{out,i}^M x_i^M - w_{iT}^M \geq 0 , \quad (25)$$

then

$$\{x : J_i^{2*}(x) \geq 0\} = \{x : -x_i + x_i^M \geq 0\} . \quad (26)$$

*Proof.* We proceed in the same way as in the previous lemma. In this case,

$$H_i^{2*} \left( x, \frac{\partial J_i^{2*}(x, 0)}{\partial x} \right)$$

writes as

$$-Q_{i-1,i}x_{i-1} + Q_{i,i+1}x_i + Q_{out,i}x_i + Q_{i,i-1}x_i - Q_{i+1,i}x_{i+1} - u_i - w_{Ti} ,$$

which immediately yields

$$H_i^{2*} \left( x, \frac{\partial J_i^{2*}(x, 0)}{\partial x} \right) \geq V_i^{-1} [Q_{out,i}x_i - w_{Ti}^M]$$

being  $x_{i-1}, x_i, x_{i+1} > 0$ ,  $Q_{i,i+1}, Q_{i,i-1} \geq 0$ .  $J_i^{2*}(x, 0) \geq 0$  then yields  $x_i \leq x_i^M$ . This and (17) imply  $J_i^{2*}(x, t) \geq 0$  for all  $t \geq 0$  and  $J_i^{2*}(x) \geq 0$ . The thesis is then immediately concluded. It turns out that

$$F_i = \bigcap_{j=1}^2 \{x : J_i^{j*}(x) \geq 0\} ,$$

that is the first part of the thesis. That  $F_i$  is invariant under the action of the controller  $i$  is also easily verified by checking that, thanks to the definition of the controller and (16), (17), the velocity vector always points inward  $F_i$  whenever  $x_i$  is on the boundary of  $F_i$ . This also points out that at each time Zone  $i$  is either cooling down or heating up. To prove the last part of the proposition, we focus on each one of these mutually exclusive cases.

*Zone  $i$  is cooling down.* In this case, the computation of the maximizing controller  $U(\cdot)$  for the game  $J_i^{2*}(x, t)$  reduces to the optimization problem

$$\max_{U \in \mathbf{U}} \min_{W \in \mathbf{W}} \{-Q_{i-1,i}x_{i-1} + Q_{i,i+1}x_i + Q_{out,i}x_i + Q_{i,i-1}x_i - Q_{i+1,i}x_{i+1} - u_i - w_{Ti}\} .$$

This admits the solution

$$Q_{i-1,i}^* = Q_{i+1,i}^* = 0, \quad Q_{out,i}^* = Q_{out,i}^M, \quad u_i^* = 0, \quad w_{Ti*} = w_{Ti}^M,$$

where  $Q_{i,i+1}^*$ ,  $Q_{i,i-1}^*$ ,  $Q_{in,i}^*$  are the largest values which satisfy

$$Q_{i,i+1}^* + Q_{out,i}^M + Q_{i,i-1}^* = Q_{in,i}^*,$$

and comply to the constraints imposed by the neighbors, embodied by the logic variables  $\sigma_{i,i+1}^{(i+1)}$ ,  $\sigma_{i+1,i}^{(i+1)}$ ,  $\sigma_{i-1,i}^{(i-1)}$ ,  $\sigma_{i,i-1}^{(i-1)}$ . Such largest values are achieved by letting

$$Q_{in,i}^* = Q_{in,i}^M, \quad \sigma_{i,i+1}^{(i)*} = 1, \quad \sigma_{i,i-1}^{(i)*} = 1.$$

In fact, by condition (15), the choice above yields that

$$Q_{i,i+1}^* = \begin{cases} \sum_{j=1}^{\gamma_{i+1}^+ + 1} Q_{out,j+i}^M & \text{for } \sigma_{i,i+1}^{(i+1)} = 1 \\ 0 & \text{for } \sigma_{i,i+1}^{(i+1)} = 0 \end{cases}$$

and

$$Q_{i,i-1}^* = \begin{cases} \sum_{j=1}^{\gamma_{i-1}^- + 1} Q_{out,i-j}^M & \text{for } \sigma_{i,i-1}^{(i-1)} = 1 \\ 0 & \text{for } \sigma_{i,i-1}^{(i-1)} = 0. \end{cases}$$

*Zone i is heating up.* The optimization problem in this case takes the form

$$\max_{U \in \mathbf{U}} \min_{W \in \mathbf{W}} \{Q_{i-1,i}(x_{i-1} - x_i) + Q_{i+1,i}(x_{i+1} - x_i) - Q_{in,i}x_i + u_i + w_{Ti}\}.$$

We have

$$Q_{in,i}^* = 0, \quad u_i^* = u_i^M, \quad w_{Ti*} = w_{Ti}^m.$$

The optimal values for the remaining control variables can be decided on the basis of the values of  $x_{i-1}$ ,  $x_i$ ,  $x_{i+1}$  and of the coordinating logic variables set by the neighbors. The cases to be examined are as follows.

- $x_{i-1} \leq x_i \wedge x_i \geq x_{i+1}$ . In this case  $Q_{i-1,i}^* = Q_{i+1,i}^* = 0$ . By (3), and  $Q_{in,i}^* = 0$  established above, it must also be true  $Q_{out,i}^* = Q_{i,i-1}^* = Q_{i,i+1}^* = 0$ . Hence, we set

$$\sigma_{i,i-1}^{(i)*} = \sigma_{i-1,i}^{(i)*} = \sigma_{i+1,i}^{(i)*} = \sigma_{i,i+1}^{(i)*} = 0.$$

This forbids neighbors to draw (respectively, release) air from (to) Zone  $i$ .

- $x_{i-1} > x_i \wedge x_i \geq x_{i+1}$ . We must have  $Q_{i+1,i}^* = 0$ , and therefore, by the flow balance,

$$Q_{i-1,i}^* = Q_{out,i}^* + Q_{i,i-1}^* + Q_{i,i+1}^*, \tag{27}$$

where all the values in the equation above must be as large as possible (so that, by (2),  $Q_{i,i-1}^* = 0$ ) and conform to the constraints imposed by Zone  $i-1$ .

If  $\sigma_{i-1,i}^{(i-1)} = 0$ , then

$$Q_{i-1,i}^* = Q_{out,i}^* = Q_{i,i-1}^* = Q_{i,i+1}^* = 0,$$

and we must set

$$\sigma_{i,i-1}^{(i)*} = \sigma_{i+1,i}^{(i)*} = \sigma_{i,i+1}^{(i)*} = 0.$$

In this case, it is not necessary to set  $\sigma_{i-1,i}^{(i-1)*} = 0$ , because  $\sigma_{i-1,i}^{(i-1)} = 0$  already. In fact, we set  $\sigma_{i-1,i}^{(i)*} = 1$ , for, if in the meanwhile Zone  $i-1$  happens to switch to a new controller,  $\sigma_{i-1,i}^{(i)} = 1$  signals to Zone  $i-1$  that Zone  $i$  is willing to draw warm air from it, and this will correctly affect the decision of Zone  $i-1$  regarding which controller to switch.

On the other hand, if  $\sigma_{i-1,i}^{(i-1)} = 1$ , by (27), we must set

$$\sigma_{i-1,i}^{(i)*} = 1, \quad Q_{out,i}^* = Q_{out,i}^M, \quad \sigma_{i,i+1}^{(i)*} = 1.$$

In this way,

$$Q_{i,i+1}^* = \begin{cases} \sum_{j=1}^{\gamma_{i+1}^+ + 1} Q_{out,j+i}^M & \text{for } \sigma_{i,i+1}^{(i+1)} = 1 \\ 0 & \text{for } \sigma_{i,i+1}^{(i+1)} = 0 \end{cases}$$

and

$$Q_{i-1,i}^* = Q_{out,i}^M + Q_{i,i+1}^*,$$

is at its maximum given the constraints.

- $x_{i-1} \leq x_i \wedge x_i < x_{i+1}$ . It is the case symmetric to the previous one and the optimal solution can be immediately derived. If  $\sigma_{i+1,i}^{(i+1)} = 0$ , then

$$Q_{out,i}^* = \sigma_{i,i-1}^{(i)*} = \sigma_{i-1,i}^{(i)*} = \sigma_{i,i+1}^{(i)*} = 0.$$

and  $\sigma_{i+1,i}^{(i)*} = 1$ . If  $\sigma_{i+1,i}^{(i+1)} = 1$ , then

$$\sigma_{i,i-1}^{(i)*} = 1, \quad Q_{out,i}^* = Q_{out,i}^M, \quad \sigma_{i+1,i}^{(i)*} = 1.$$

- $x_{i-1} > x_i \wedge x_i < x_{i+1}$ . We distinguish 5 sub-cases:

(i)  $\sigma_{i-1,i}^{(i-1)} = 0 \wedge \sigma_{i+1,i}^{(i+1)} = 0$ . We must have

$$Q_{i-1,i}^* = Q_{i+1,i}^* = 0,$$

and hence

$$Q_{i,i-1}^* = Q_{out,i}^* = Q_{i,i+1}^* = 0$$

as well. To enforce this, we set

$$\sigma_{i,i-1}^{(i)*} = \sigma_{i,i+1}^{(i)*} = 0.$$

Even in this case, it is not necessary to set  $\sigma_{i-1,i}^{(i)*} = \sigma_{i+1,i}^{(i)*} = 0$ , for  $\sigma_{i-1,i}^{(i-1)} = \sigma_{i+1,i}^{(i+1)} = 0$ , and indeed we set

$$\sigma_{i-1,i}^{(i)*} = \sigma_{i+1,i}^{(i)*} = 1.$$

(ii)  $\sigma_{i-1,i}^{(i-1)} = 1 \wedge \sigma_{i+1,i}^{(i+1)} = 0$ . This reduces to the case

$$x_{i-1} > x_i \wedge x_i \geq x_{i+1} \wedge \sigma_{i-1,i}^{(i-1)} = 1,$$

already examined above.

(iii)  $\sigma_{i-1,i}^{(i-1)} = 0 \wedge \sigma_{i+1,i}^{(i+1)} = 1$ . This reduces to the case

$$x_{i-1} \leq x_i \wedge x_i < x_{i+1} \wedge \sigma_{i+1,i}^{(i+1)} = 1.$$

(iv)  $\sigma_{i-1,i}^{(i-1)} = 1 \wedge \sigma_{i+1,i}^{(i+1)} = 1 \wedge x_{i-1} \geq x_{i+1}$ . As for (ii), this reduces to the case

$$x_{i-1} > x_i \wedge x_i \geq x_{i+1} \wedge \sigma_{i-1,i}^{(i-1)} = 1.$$

(v)  $\sigma_{i-1,i}^{(i-1)} = 1 \wedge \sigma_{i+1,i}^{(i+1)} = 1 \wedge x_{i-1} < x_{i+1}$ . As for (iii), this case reduces to

$$x_{i-1} \leq x_i \wedge x_i < x_{i+1} \wedge \sigma_{i+1,i}^{(i+1)} = 1.$$

## B Proof of Lemma 1

We consider combinations of modes for Zone 1 and 2. Note that the majority possible cases are not feasible, and hence the analysis is simpler than it could appear. We shall refer to the case in which Zone 1 is in Mode i and Zone 2 is in Mode j as *Case i.j*. For the convenience of the readers, while the first cases will be examined more in detail, we shall proceed faster as we get acquainted with the line of reasoning which underlies the proof. Let us recall that we are to prove that the flow balance (3) is fulfilled for  $i = 1, 2$ , i.e. for Zone 1 and 2, provided that:

- If Zone 2 is in Mode 1, 4 or 8,

$$Q_{23} = \sum_{j=1}^{\gamma_2^+} Q_{out,j+2}^M, \quad (28)$$

with  $\gamma_2^+ = (\gamma_3^+ + 1)\sigma_{23}^{(3)}$ , and  $Q_{32} = 0$ .



- If Zone 2 is in Mode 2, 3, 5, 7 or 10,

$$Q_{23} = 0 \text{ and } Q_{32} = 0. \quad (29)$$

- If Zone 2 is in Mode 6, 9 or 11,  $Q_{23} = 0$  and

$$Q_{32} = \sum_{j=1}^{\gamma_3^-} Q_{out,3-j}^M, \quad (30)$$

with  $\gamma_3^- = \gamma_2^- + 1$ .

Consider the *Case 1.1*. Then both zones are cooling down and no air exchange is possible. Indeed,

$$\gamma_1^+ = (\gamma_2^+ + 1)\sigma_{12}^{(1)}\sigma_{12}^{(2)} = (\gamma_2^+ + 1) \cdot 1 \cdot 0 = 0 \text{ and } \gamma_2^- = \sigma_{21}^{(1)}\sigma_{21}^{(2)} = 0 \cdot 1 = 0.$$

As a consequence,  $Q_{in,1} = Q_{out,1}^M$ . As far as Zone 2 is concerned, observe that  $Q_{out,2} = Q_{out,2}^M$  and  $\gamma_2^+ = (\gamma_3^+ + 1) \cdot 1 \cdot \sigma_{23}^{(3)}$ . Hence,

$$Q_{in,2} = Q_{out,2}^M + \sum_{j=1}^{\gamma_2^-} Q_{out,2-j}^M + \sum_{j=1}^{\gamma_2^+} Q_{out,j+2}^M = Q_{out,2}^M + \sum_{j=1}^{\gamma_2^+} Q_{out,j+2}^M.$$

We conclude that, as a total of  $Q_{out,1}^M + Q_{out,2}^M$  is removed from Zone 1 and 2, and a total inflow of  $Q_{out,1}^M + Q_{out,2}^M + \sum_{j=1}^{\gamma_2^+} Q_{out,j+2}^M$  is allowed through the 2 inlets, the flow balance for the 2 zones is fulfilled provided that (28) holds.

*Case 2.1.* Mode 2 for Zone 1 imposes that  $Q_{out,1} = Q_{in,1} = 0$ . As before,  $\gamma_2^- = 0$ ,  $\gamma_2^+ = (\gamma_3^+ + 1)\sigma_{23}^{(3)}$ ,  $Q_{out,2} = Q_{out,2}^M$  and  $Q_{in,2} = Q_{out,2}^M + \sum_{j=1}^{\gamma_2^+} Q_{out,j+2}^M$ , we can draw exactly the same conclusion as above. The *Case 3.1* is not feasible because, Zone 2 in Mode 1 imposes  $\sigma_{21}^{(2)} = 1$ , and this induces an immediate transition of Zone 1 from Mode 3 to 4. On the other hand, it is admissible the *Case 4.1*. As  $Q_{out,1} = Q_{out,1}^M$  and  $Q_{in,1} = 0$ , then  $Q_{21} = Q_{out,1}^M$ . Consistently,  $\gamma_2^- = \sigma_{21}^{(1)}\sigma_{21}^{(2)} = 1$ . Additionally,  $\gamma_2^+ = (\gamma_3^+ + 1) \cdot 1 \cdot \sigma_{23}^{(3)}$  yields

$$Q_{in,2} = Q_{out,2}^M + \sum_{j=1}^{\gamma_2^-} Q_{out,2-j}^M + \sum_{j=1}^{\gamma_2^+} Q_{out,j+2}^M = Q_{out,2}^M + Q_{out,1}^M + \sum_{j=1}^{\gamma_2^+} Q_{out,j+2}^M,$$

that is, again, the flow balance for the 2 zones is fulfilled provided that (28) is true.

*Case 1.2.*  $\gamma_1^+ = (\gamma_2^+ + 1)\sigma_{12}^{(1)}\sigma_{12}^{(2)} = (\gamma_2^+ + 1) \cdot 1 \cdot 0 = 0$  yields  $Q_{in,1} = Q_{out,1}^M$ , with  $Q_{out,1} = Q_{out,1}^M$ . In Mode 2,  $Q_{in,2} = Q_{out,2} = 0$ , and  $\gamma_2^- = \gamma_2^+ = 0$ , therefore for the 2 zones to have the flow balance fulfilled, (29) must be true. Under *Cases 2.2* and *3.2* (the former is feasible only if  $x_1 = x_2$ ),  $Q_{in,i} = Q_{out,i} = 0$  for both  $i = 1, 2$ . Hence, for the flow balance in the 2 zones to be fulfilled, (29) must hold. The *Case 4.2* is not feasible, as Zone 1 being in Mode 4 requires  $\sigma_{21}^{(2)} = 1$  (otherwise, a transition to Mode 3 would occur), while Zone 2 in Mode 2 imposes  $\sigma_{21}^{(2)} = 0$ .

*Case 1.3.* This is unfeasible, for Mode 1 imposes  $\sigma_{12}^{(1)} = 1$ , which would imply a transition out from Mode 3 for Zone 2. Under the *Case 2.3*,  $Q_{in,i} = Q_{out,i} = 0$  for both  $i = 1, 2$ , and hence the flow balance for Zones 1 and 2 is fulfilled provided that (29) holds. The two *Case 3.3* and *Case 4.3* are not compatible, as it can not be simultaneously  $x_1 > x_2$  and  $x_1 < x_2$ .

When Zone 2 is in Mode 4, only the *Case 1.4* is possible. We have:  $Q_{out,i} = Q_{out,i}^M$ , for  $i = 1, 2$  and  $Q_{in,2} = 0$ . Also,  $\gamma_1^+ = \gamma_2^+ + 1$ , and  $\gamma_2^+ = (\gamma_3^+ + 1) \cdot 1 \cdot \sigma_{23}^{(3)}$ , imply  $Q_{in,1} = Q_{out,1} + Q_{out,2} + \sum_{j=1}^{\gamma_2^+} Q_{out,j+2}^M$ , from which we conclude that the flow balance is preserved provided that  $Q_{23} = \sum_{j=1}^{\gamma_2^+} Q_{out,j+2}^M$ .

When Zone 2 is in Mode 5, only *Cases 1.5* and *2.5-3.5* must be checked. In the former case,  $\gamma_1^+ = 0$ , and hence  $Q_{in,1} = Q_{out,1} = Q_{out,1}^M$ . On the other hand  $Q_{in,2} = Q_{out,2} = 0$ , and hence (29) must be true. In the latter two cases,  $Q_{in,i} = Q_{out,i} = 0$  for both  $i = 1, 2$ , and again (29) must be true.

Consider now the *Case 1.6*. We have  $Q_{out,1} = Q_{out,1}^M$ ,  $\gamma_1^+ = (\gamma_2^+ + 1)\sigma_{12}^{(1)}\sigma_{12}^{(2)} = 0$ , and therefore  $Q_{in,1} = Q_{out,1}^M$ . On the other hand,  $Q_{out,2} = Q_{out,2}^M$ ,  $Q_{in,2} = 0$ , and  $\gamma_2^+ = \gamma_2^- = 0$ , whereas  $\gamma_3^- = \sigma_{32}^{(2)}\sigma_{32}^{(3)} = 1$ , since for Zone 2 to be in Mode 6, it is required  $\sigma_{32}^{(3)} = 1$ . We conclude that (30) must be true. Under *Case 2.6* (which is feasible only if  $x_1 = x_2$ ), it is easily verified that  $Q_{32} = Q_{out,1}^M$ ,  $\gamma_2^- = 0$  and  $\gamma_3^- = 1$ . Hence,  $Q_{32} = \sum_{j=1}^{\gamma_3^-} Q_{out,3-j}^M$ , that is (30). *Case 3.6* is not feasible. We are left with *Case 4.6*. In this case,  $Q_{in,i} = 0$  and  $Q_{out,i} = Q_{out,i}^M$  for  $i = 1, 2$ , so that the fulfillment of the flow balance requires  $Q_{32} = Q_{out,1}^M + Q_{out,2}^M$ . As  $\gamma_2^- = 1$  and  $\gamma_3^- = (\gamma_2^- + 1) = 2$ , (30) is

immediately proven.

We examine now the *Case 1.7*. This is not feasible, for Mode 1 imposes  $\sigma_{12}^{(1)} = 1$ , which would cause Zone 2 to switch from Mode 7 to Mode 8. We discard the *Cases 3.7* and *4.7* too, as the requirements on the temperatures  $x_1$ ,  $x_2$  are contradictory. This is still true for all the cases for which Zone 1 is in Mode 3 or 4 whereas Zone 2 is in mode 7 to 11, and these will be ignored in the sequel. The only case to investigate is *Case 2.7*. It is immediately verified that (29) must hold.

When Zone 2 is in Mode 8, the only cases to consider are *Cases 1.8* and *2.8*. Under *Case 1.8*,  $Q_{out,i} = Q_{out,i}^M$ , for  $i = 1, 2$ ,  $Q_{in,1} = Q_{in,1}^M$ ,  $Q_{in,2} = 0$ ,  $\gamma_1^+ = \gamma_2^+ + 1$ , and  $\gamma_2^+ = (\gamma_3^+ + 1) \cdot 1 \cdot \sigma_{23}^{(3)}$ . Hence  $Q_{in,1} = Q_{out,1}^M + \sum_{j=1}^{\gamma_2^++1} Q_{out,j+1}^M = Q_{out,1}^M + Q_{out,2}^M + \sum_{j=1}^{\gamma_2^++1} Q_{out,j+2}^M$ , which gives (28). *Case 2.8* yields  $\gamma_1^+ = \gamma_2^- = 0$ ,  $Q_{out,2} = Q_{out,2}^M$ ,  $Q_{in,2} = 0$ , and  $\gamma_2^+ = (\gamma_3^+ + 1) \sigma_{23}^{(3)}$ , so that (28) must hold.

The *Case 1.9* is not feasible because there are opposing requests on  $\sigma_{12}^{(1)}$ . We examine the only possible case, namely *Case 2.9*. We have  $\gamma_1^+ = 0$ ,  $Q_{out,1} = Q_{in,1} = 0$ ,  $Q_{out,2} = Q_{out,2}^M$ ,  $Q_{in,2} = 0$ ,  $\gamma_2^- = 0$ ,  $\gamma_3^- = 1$ . The latter 2 equalities in particular allow to verify (30).

In the *Case 1.10*,  $Q_{out,1} = Q_{out,1}^M$ ,  $Q_{in,1} = Q_{in,1}^M$ , with  $Q_{in,1}^M$  easily computed from  $\gamma_1^+ = (\gamma_2^+ + 1)$ ,  $\gamma_2^+ = (\gamma_3^+ + 1) \sigma_{23}^{(3)}$ . Moreover,  $Q_{out,2} = Q_{out,2}^M$ ,  $Q_{in,2} = 0$ , and as a result the correctness of (29) is proven. The *Case 2.10* is not feasible (compare the requirements on  $\sigma_{12}^{(1)}$  by the 2 Zones).

We are left with the last feasible case, i.e. *Case 1.11*. Then  $Q_{out,1} = Q_{out,1}^M$  and  $\gamma_1^+ = 0$ , so that  $Q_{in,1} = Q_{out,1}^M$ . Moreover,  $Q_{out,2} = Q_{out,2}^M$ ,  $Q_{in,2} = 0$ ,  $\gamma_2^- = 0$ ,  $\gamma_2^+ = 0$ , and  $\gamma_3^- = 1$ . The validity of (30) is immediately inferred.

## C Proof of Lemma 2

We prove that Zones  $1, 2, \dots, i+1$  conditionally satisfy the flow balance, that is, the flow balance is fulfilled for those zones provided that:

- If Zone  $i+1$  is in Mode 1, 4 or 8,

$$Q_{i+1,i+2} = \sum_{j=1}^{\gamma_{i+1}^+} Q_{out,j+i+1}^M, \quad (31)$$

with  $\gamma_{i+1}^+ = (\gamma_{i+2}^+ + 1) \sigma_{i+1,i+2}^{(i+2)}$ , and  $Q_{i+2,i+1} = 0$ .

- If Zone  $i+1$  is in Mode 2, 3, 5, 7 or 10,

$$Q_{i+1,i+2} = 0 \text{ and } Q_{i+2,i+1} = 0. \quad (32)$$

- If Zone  $i+1$  is in Mode 6, 9 or 11,  $Q_{i+1,i+2} = 0$  and

$$Q_{i+2,i+1} = \sum_{j=1}^{\gamma_{i+2}^-} Q_{out,i+2-j}^M, \quad (33)$$

with  $\gamma_{i+2}^- = \gamma_{i+1}^- + 1$ .

In principal the proof goes very similar to the one of the previous lemma, but we change it to cope with the fact that in the current case the controller mode combinations occur in larger number. Zone  $i+1$  depends on Zone  $i$  and  $i+2$  so in total there are  $11 \cdot 11 \cdot 11 = 1331$  different modes to consider. Many of these modes are however not feasible i.e. specific mode combinations would immediately cause a transition to a different mode combination. To narrow the number of combinations to consider, it suffices to take into account the possible combinations of coordinating logic variables for Zones  $i$ ,  $i+1$  and  $i+2$ . Noticing that simultaneous flow into a zone from both neighboring zones is impossible, it is enough to consider the following cases:

1.

$$\begin{aligned} \sigma_{i,i+1}^{(i)} \sigma_{i,i+1}^{(i+1)} &= 0 & \sigma_{i+1,i+2}^{(i+1)} \sigma_{i+1,i+2}^{(i+2)} &= 0 \\ \sigma_{i+1,i}^{(i)} \sigma_{i+1,i}^{(i+1)} &= 0 & \sigma_{i+2,i+1}^{(i+1)} \sigma_{i+2,i+1}^{(i+2)} &= 0 \end{aligned} \quad (34)$$

2.

$$\begin{aligned} \sigma_{i,i+1}^{(i)} \sigma_{i,i+1}^{(i+1)} &= 0 & \sigma_{i+1,i+2}^{(i+1)} \sigma_{i+1,i+2}^{(i+2)} &= 0 \\ \sigma_{i+1,i}^{(i)} \sigma_{i+1,i}^{(i+1)} &= 1 & \sigma_{i+2,i+1}^{(i+1)} \sigma_{i+2,i+1}^{(i+2)} &= 0 \end{aligned} \quad (35)$$

3.

$$\begin{aligned} \sigma_{i,i+1}^{(i)} \sigma_{i,i+1}^{(i+1)} &= 0 & \sigma_{i+1,i+2}^{(i+1)} \sigma_{i+1,i+2}^{(i+2)} &= 1 \\ \sigma_{i+1,i}^{(i)} \sigma_{i+1,i}^{(i+1)} &= 0 & \sigma_{i+2,i+1}^{(i+1)} \sigma_{i+2,i+1}^{(i+2)} &= 0 \end{aligned} \quad (36)$$

4.

$$\begin{aligned} \sigma_{i,i+1}^{(i)} \sigma_{i,i+1}^{(i+1)} &= 0 & \sigma_{i+1,i+2}^{(i+1)} \sigma_{i+1,i+2}^{(i+2)} &= 1 \\ \sigma_{i+1,i}^{(i)} \sigma_{i+1,i}^{(i+1)} &= 1 & \sigma_{i+2,i+1}^{(i+1)} \sigma_{i+2,i+1}^{(i+2)} &= 0 \end{aligned} \quad (37)$$

5.

$$\begin{aligned} \sigma_{i,i+1}^{(i)} \sigma_{i,i+1}^{(i+1)} &= 1 & \sigma_{i+1,i+2}^{(i+1)} \sigma_{i+1,i+2}^{(i+2)} &= 0 \\ \sigma_{i+1,i}^{(i)} \sigma_{i+1,i}^{(i+1)} &= 0 & \sigma_{i+2,i+1}^{(i+1)} \sigma_{i+2,i+1}^{(i+2)} &= 0 \end{aligned} \quad (38)$$

6.

$$\begin{aligned} \sigma_{i,i+1}^{(i)} \sigma_{i,i+1}^{(i+1)} &= 0 & \sigma_{i+1,i+2}^{(i+1)} \sigma_{i+1,i+2}^{(i+2)} &= 0 \\ \sigma_{i+1,i}^{(i)} \sigma_{i+1,i}^{(i+1)} &= 0 & \sigma_{i+2,i+1}^{(i+1)} \sigma_{i+2,i+1}^{(i+2)} &= 1 \end{aligned} \quad (39)$$

7.

$$\begin{aligned} \sigma_{i,i+1}^{(i)} \sigma_{i,i+1}^{(i+1)} &= 1 & \sigma_{i+1,i+2}^{(i+1)} \sigma_{i+1,i+2}^{(i+2)} &= 1 \\ \sigma_{i+1,i}^{(i)} \sigma_{i+1,i}^{(i+1)} &= 0 & \sigma_{i+2,i+1}^{(i+1)} \sigma_{i+2,i+1}^{(i+2)} &= 0 \end{aligned} \quad (40)$$

8.

$$\begin{aligned} \sigma_{i,i+1}^{(i)} \sigma_{i,i+1}^{(i+1)} &= 0 & \sigma_{i+1,i+2}^{(i+1)} \sigma_{i+1,i+2}^{(i+2)} &= 0 \\ \sigma_{i+1,i}^{(i)} \sigma_{i+1,i}^{(i+1)} &= 1 & \sigma_{i+2,i+1}^{(i+1)} \sigma_{i+2,i+1}^{(i+2)} &= 1 \end{aligned} \quad (41)$$

*Case (i).* In this case, it is immediately seen that  $\gamma_i^+ = \gamma_i^- = 0$  and  $\gamma_{i+1}^+ = \gamma_{i+1}^- = \gamma_{i+2}^- = 0$ . The former pair of equalities and the hypothesis that Zone  $1, 2, \dots, i$  conditionally satisfy the flow balance yield that necessarily  $Q_{i,i+1} = Q_{i+1,i} = 0$ . We distinguish two sub cases: Zone  $i+1$  is heating up and Zone  $i+1$  is cooling down. If Zone  $i+1$  is heating up, then  $Q_{in,i+1} = 0$ , and since no internal ventilation can occur with neighboring zones, necessarily  $Q_{out,i+1} = 0$  as well. This implies  $Q_{i+1,i+2} = Q_{i+2,i+1} = 0$ . When zone  $i+1$  is cooling down the outflow provided by the fan is  $Q_{out,i+1} = Q_{out,i+1}^M$ . Now (34) leads to:

$$Q_{in,i+1} = Q_{out,i+1}^M + \sum_{j=1}^{\gamma_{i+1}^-} Q_{out,i+1-j}^M + \sum_{j=1}^{\gamma_{i+1}^+} Q_{out,j+i+1}^M = Q_{out,i+1}^M \quad (42)$$

and hence  $Q_{i+1,i+2} = Q_{i+2,i+1} = 0$ . The facts above prove that (31) and (32) are true. We are left with verifying that Zone  $i+1$  can not be in Mode 6, 9, or 11, as this would require  $Q_{i+2,i+1} \neq 0$  (see (33)). Zone  $i+1$  can be in Mode 6, 9, or 11 only if  $\sigma_{i+2,i+1}^{(i+1)} \sigma_{i+2,i+1}^{(i+2)} = 1$ , which contradicts (34). *Case (ii).* It is easily verified that in this case Zone  $i$  can be in Mode 6, 9 or 11 only, whereas Zone  $i+1$  can be only in Mode 1. By hypothesis, the former fact gives that  $Q_{i+1,i} = \sum_{j=1}^{\gamma_{i+1}^-} Q_{out,i+1-j}^M$ . Because of (35),  $\sum_{j=1}^{\gamma_{i+1}^+} Q_{out,j+i+1}^M = 0$ , and the inlet is set to:

$$Q_{in,i+1} = Q_{out,i+1}^M + \sum_{j=1}^{\gamma_{i+1}^-} Q_{out,i+1-j}^M$$

Since zone  $i+1$  is cooling down with  $Q_{out,i+1} = Q_{out,i+1}^M$ , this gives  $Q_{i+1,i+2} = Q_{i+2,i+1} = 0$ . As for the case before, (31) and (32) trivially hold, while Zone  $i+1$  can never be in Mode 6, 9, or 11. *Case (iii).* With (36), no air exchange between Zone  $i$  and  $i+1$  is possible, and in particular Zone  $i+1$  is in Mode 1, i.e. cooling down with  $Q_{out,i+1} = Q_{out,i+1}^M$ . The inlet is set to:

$$Q_{in,i+1} = Q_{out,i+1}^M + \sum_{j=1}^{\gamma_{i+1}^-} Q_{out,i+1-j}^M + \sum_{j=1}^{\gamma_{i+1}^+} Q_{out,j+i+1}^M = Q_{out,i+1}^M + \sum_{j=1}^{\gamma_{i+1}^+} Q_{out,j+i+1}^M$$

It is promptly verified that (31) holds true. *Case (iv).* As before Zone  $i+1$  is operating at Mode 1, with the inflow

set to:

$$Q_{in,i+1} = Q_{out,i+1}^M + \sum_{j=1}^{\gamma_{i+1}^-} Q_{out,i+1-j}^M + \sum_{j=1}^{\gamma_{i+1}^+} Q_{out,j+i+1}^M$$

Zone  $i$  fulfills (20) because it can be in Mode 6, 9 or 11, and hence (31) holds. *Case (v)* In this case, Zone  $i$  can operate in Modes 1,4 or 8, whereas Zone  $i+1$  in Mode 4, 8 or 10. By hypothesis, the former gives  $Q_{i,i+1} = \sum_{j=1}^{\gamma_i^+} Q_{out,i+j}^M = Q_{out,i+1}^M$ , as  $\gamma_{i+1}^+ = 0$ . Zone  $i+1$  in Mode 4, 8 or 10 gives  $Q_{out,i+1} = Q_{out,i+1}^M$  and  $Q_{in,i+1} = 0$ , so that  $Q_{i+1,i+2} = Q_{i+2,i+1} = 0$ , which is precisely (31) or (32). *Case (vi)*. From (39), no air is exchanged between zone  $i$  and  $i+1$ . Furthermore,  $\sigma_{i+2,i+1}^{(i+1)} \cdot \sigma_{i+2,i+1}^{(i+2)} = 1$  gives that Zone  $i+1$  can be in Mode 6, 9, 10 or 11. At a second sight, we can rule out Mode 10, as Zone  $i+1$  is in Mode 10 only if  $\sigma_{i,i+1}^{(i)} \sigma_{i,i+1}^{(i+1)} = 1$ , which contradicts (39). Zone  $i+1$  being in one of these feasible modes gives  $Q_{i+2,i+1} = Q_{out,i+1}^M$ , which proves (33), as  $\gamma_{i+2}^- = 1$ . *Case (vii)*. Here air flows from zone  $i$  to  $i+1$  and from  $i+1$  to  $i+2$ . In particular, Zone  $i$  can be in either one of Modes 1,4, 8, whereas Zone  $i+1$  can be either in Mode 4 or 10, with  $Q_{out,i+1} = Q_{out,i+1}^M$ . Being  $\gamma_i^+ = \gamma_{i+1}^+ + 1$ , we conclude that

$$Q_{i+1,i+2} = \sum_{j=1}^{\gamma_{i+1}^+} Q_{out,i+1+j}^M$$

with  $\gamma_{i+1}^+ = (\gamma_{i+2}^+ + 1) \sigma_{i+1,i+2}^{(i+1)} \sigma_{i+1,i+2}^{(i+2)} = \gamma_{i+2}^+ + 1$ , that is (31). *Case (viii)*. This is the exact opposite case of the previous one. Zone  $i$  is in Mode 6, 9, or 11, with  $\gamma_{i+1}^- = \gamma_i^- + 1$ . Zone  $i+1$  is heating up with warm air from zone  $i+2$  so it operates the fan at its maximum  $Q_{out,i+1}^M$ . In particular, it is in Mode 6 or 9, which implies that (33) is fulfilled. This concludes the proof.